Numerical Solution of a Generalized Wahba Problem for a Spinning Spacecraft

Mark L. Psiaki∗ and Joanna C. Hinks#
Cornell University, Ithaca, N.Y. 14853-7501

A numerical solution algorithm has been developed to solve a generalization of Wahba's attitude determination problem to the case of unknown attitude and attitude rate. It provides a robust global solution to nonlinear batch attitude and rate estimation problems for spinning spacecraft. The original Wahba problem seeks the attitude that fits a set of unit-normalized direction vector measurements. The generalized problem seeks an initial attitude and rate that, when combined with a torque-free rigid-body attitude dynamics model, fit a time series of direction vector measurements. The new algorithm combines an inner analytic solution for the unknown initial attitude quaternion with an outer numerical solution for the unknown initial rate. The inner problem can be cast in a standard Wahba form, and it is solved in closed form using an adaptation of the q method. The outer problem relies in a trust-region implementation of Newton's method in order to ensure global convergence. The global solution is determined by re-solving starting from a random set of first guesses of the attitude rate that cover the space of non-aliasing rates. Tests on truth-model simulation data for minor-axis and major-axis spinning spacecraft confirm the new algorithm's ability to achieve global convergence and its estimation accuracy's consistency with its computed estimation error covariance matrix.

I. Introduction

Many spacecraft missions require an ability to estimate the attitude of the spacecraft, and many methods exist for attitude determination based on sensor data1. Various kinds of sensor data can be used, e.g., from a magnetometer, a sun sensor, a star camera, a horizon-crossing indicator, and a 3-axis rate gyro2. Various kinds of algorithms can be used to process the data ranging from point-wise batch filters3 to sigma-points dynamic filters4.

All 3-axis attitude estimation problems are inherently nonlinear. Various techniques exist to deal with general nonlinear estimation4,5 and with the particular nonlinearities associated with attitude estimation1,3,4,6. As yet, however, there is no one algorithm that is guaranteed to find the globally optimal attitude estimate for all possible nonlinear attitude determination problems.

There is a special attitude determination problem for which a globally optimal nonlinear solution exists: Wahba's problem7. Wahba's problem seeks the attitude representation that transforms two or more unit direction vectors from reference coordinates to spacecraft body coordinates with the minimum weighted sum of squared errors. It is defined using vectors measured all at the same point of time or measured over an interval when the spacecraft does not rotate with respect to reference coordinates. The globally optimal solution is known as the q method. It determines the optimal attitude quaternion by solving an eigenvalue problem for a 4x4 symmetric matrix2,3. This solution can be used to determine the attitude based on measurements from certain types of sun sensors, from a 3-axis magnetometer, or from individual stars in a star camera field of view. The solution is unique if 2 or more of the measured attitude vectors are linearly independent.

The robustness of the nonlinear Wahba solution has led to various attempts to extend it to deal with time-varying situations, in which the spacecraft can rotate with respect to reference coordinates between vector measurements6,8,9,10,11,12. These attempts solve problems with varying degrees of generality, and they have met with varying degrees of success. Some require additional sensors such as rate-gyros, e.g., Refs. 6, 9, and 10. Others require an Euler dynamics model of the spacecraft attitude rate time history11,12.

∗ Professor, Sibley School of Mechanical and Aerospace Engineering. Associate Fellow, AIAA.
# Graduate student, Sibley School of Mechanical and Aerospace Engineering.
The present paper seeks to extend the results of Refs. 11 and 12, which deal with a particular generalized form of Wahba's problem. That form assumes that the spacecraft undergoes torque-free, rigid-body rotation and that its moment-of-inertia matrix is known. It assumes that the only available measurements are attitude vector observations and that they are available at a sequence of times. These papers seek to solve a batch estimation problem in order to determine the initial attitude quaternion and the initial attitude rate that best fit the time series of vector measurements. Reference 11 defines this class of generalizations to Wahba's problem, including one in which the spacecraft spins about a known axis with an unknown constant rate. It presents an analytic solution to a restricted form of this latter problem, one involving only 2 vector measurements. It also suggests solution strategies for the general torque-free version of the problem, the one that includes the possibility of attitude rate time variations due to nutations. Reference 12 concentrates on solving a restricted form of the generalized Wahba problem that includes nutations. Its restricted form involves only 3 vector measurements, and it assumes axial symmetry of the spacecraft. This leads to various systems of 6 nonlinear equations in 6 unknowns or the equivalent, and Ref. 12 solves them partly via analytical techniques and partly via numerical nonlinear equation solving methods.

The present paper develops and tests an algorithm to solve the least restrictive of the generalized Wahba problems from Ref. 11, the problem that includes a time-varying attitude rate due to torque-free nutations and that allows a general moment-of-inertia matrix. It implements an approach that is suggested in Ref. 11: It solves an outer problem for the unknown initial attitude rate under the assumption that the unknown initial attitude quaternion is computed for any given initial rate estimate by solving an inner Wahba problem. The outer numerical optimization algorithm achieves robust convergence properties by employing two complimentary strategies. The first is a trust-region algorithm that ensures global convergence to a local minimum when starting from any given first guess of the attitude rate. The second strategy re-solves the problem from many different randomly-generated first guesses of the initial attitude rate within a known region of possible solutions. The final solution with the lowest cost is the global optimal estimate. The resulting algorithm is tested using data from a truth-model simulation.

This paper presents its new algorithm and simulation test results in 4 main sections plus conclusions. Section II reviews Wahba's problem and the generalized Wahba's problem of Ref. 11, the one whose solution is the subject of the current paper. Section III develops the inner/outer numerical solution procedure for the generalized Wahba problem. Section IV describes the truth-model simulation that is used to test the generalized Wahba solver. Section V presents the results of the truth-model simulation study of the new algorithm. Section VI summarizes this paper's contributions and gives its conclusions.

II. Wahba's Original Problem and a Generalized Form for a Spinning Spacecraft

A. Wahba's Original Problem and Original and Modified q-Method Solutions

Wahba's original problem solves for the direction cosines matrix $A$ that transforms reference unit direction vectors $r_i$ in inertial coordinates to closely match the corresponding measured unit direction vectors in spacecraft body coordinates $b_i$. That is, it solves the optimal estimation problem:

\begin{align}
\text{find:} & \quad A \\
\text{to minimize:} & \quad J(A) = \frac{1}{2} \sum_{i=1}^{m} \frac{1}{\sigma_i^2} [b_i - Ar_i]^T [b_i - Ar_i] \\
\text{subject to:} & \quad A^T A = I
\end{align}

where $\sigma_i$ is the attitude measurement accuracy of $b_i$.

The $q$-method solution uses the unit-normalized quaternion representation of the attitude $q$ and the formula for the attitude matrix in terms of $q$. $A(q)$, in order to transform the problem into the form:

\begin{align}
\text{find:} & \quad q \\
\text{to minimize:} & \quad J(q) = \sum_{i=1}^{m} \frac{1}{\sigma_i^2} \cdot q^T K_i q = \sum_{i=1}^{m} \frac{1}{\sigma_i^2} \cdot q^T \left( \sum_{i=1}^{m} K_i \right) q \\
\text{subject to:} & \quad q^T q = 1
\end{align}

The 4x4 $K_i$ matrices are defined as $^{1,2}$. 


\[
K_i = \frac{1}{\sigma_i^2} \begin{bmatrix}
|r_i b_i^T + b_i r_i^T - I_{3\times3} (b_i^T r_i)| \quad (b_i \times r_i) \\
(b_i \times r_i)^T 
\end{bmatrix}
\]

and the overall 4x4 \( K \) matrix equals \( \sum_{i=1}^{m} K_i \).

The global solution to problem (2a)-(2c) is an eigenvector: The optimal \( q \) is the unit-normalized eigenvector of \( K \) corresponding to its maximum eigenvalue. Solution for this eigenvector is straight-forward. The global nature of this solution guarantees that a Wahba's problem solution has none of the usual convergence problems associated with linearized or Taylor-series-based solutions, such as are obtained using an extended Kalman filter (EKF) or an unscented sigma-points Kalman filter (UKF).

For purposes of the present paper, it is convenient to develop a modified formulation the \( q \)-method cost function for Wahba's problem. This modified cost function takes the form:

\[
J(q) = \frac{1}{2} (Fq)^T Fq
\]

where

\[
F = \begin{bmatrix}
\frac{1}{\sigma_1} q_{c1}^T \\
\frac{1}{\sigma_1} q_{d1}^T \\
\frac{1}{\sigma_2} q_{c2}^T \\
\frac{1}{\sigma_2} q_{d2}^T \\
\vdots \\
\frac{1}{\sigma_m} q_{cm}^T \\
\frac{1}{\sigma_m} q_{dm}^T
\end{bmatrix}
\]

is a \((2m)x4\) matrix with \( q_{c1} \) and \( q_{d1} \) being the two orthogonal eigenvectors of the \( K_i \) matrix that are associated with its two equal minimum eigenvalues, \(-1/\sigma_i^2\). The two quaternions \( q_{c1} \) and \( q_{d1} \) are orthogonal to every \( q \) that satisfies the \( i \)th measurement equation exactly, i.e., to every \( q \) that satisfies \( b_i = A(q)r_i \). This re-formulation of the Wahba cost function and the associated \( q_{c1} \) and \( q_{d1} \) quaternions can be found in Ref. 13, albeit with different notation. The two quaternions' formulas are:

\[
q_{c1} = \begin{bmatrix}
1 \\
\sqrt{2(1-b_i^T r_i)} \\
0 \\
0
\end{bmatrix}
\]

where \( c_i \) is any 3x1 vector that is perpendicular to both \( r_i \) and \( b_i \) in the lower case of Eq. (6a).

The cost function in Eq. (4) enables one to use a singular-value decomposition (SVD) in order solve Wahba's problem. If the cost function of Eq. (2b) is replaced by that of Eq. (4), then the optimal solution to the problem defined by Eqs. (2a), (4), and (2c) is the quaternion associated with the minimum singular value of the matrix \( F \). Suppose that the SVD is \( F = U[S;0]V^T \), with \( U \) being a \((2m)x(2m)\) orthonormal matrix, with \( S \) being a 4x4 diagonal matrix with non-negative diagonal elements of decreasing magnitude as one moves down the diagonal, i.e. \( S = \text{diag}(s_{11}, s_{22}, s_{33}, s_{44}) \) with \( s_{11} \geq s_{22} \geq s_{33} \geq s_{44} \geq 0 \), and with \( V \) being a 4x4 orthonormal matrix. Then the optimal attitude quaternion \( q_{opt} \) is the 4th column of \( V \), and the minimizing cost at this optimum is \( J(q_{opt}) = \frac{1}{2}s_{44}^2 \).

**B. Generalized Wahba's Problem Statement for a Spinning Spacecraft**

Reference 11 defines two generalized Wahba problems for a spinning spacecraft. The first one is the simplest, involving only a constant unknown spin rate about a spin axis that is known \( a \) priori in body coordinates. The
second problem deals with the general case of torque-free motion of a rigid body, and it augments the unknown initial quaternion with an unknown initial attitude rate. The second problem is the one that is solved in the present paper.

This second generalized Wahba problem seeks the initial attitude quaternion and the initial attitude-rate vector that, when propagated in time, optimally fit the vector attitude measurements \( b_1, \ldots, b_m \) at the measurement times \( t_1, \ldots, t_m \). Re-using previously established notation as much as possible, this generalized problem can be stated in the following \( F \)-matrix form:

\[
\begin{align*}
\text{find:} & \quad q_0 \text{ and } \omega_0 \\
\text{to minimize:} & \quad J_{b}(q_0; \omega_0) = \frac{1}{2} [F(\omega_0)q_0]^T F(\omega_0)q_0 \\
\text{subject to:} & \quad q_0^T q_0 = 1
\end{align*}
\]

where \( q_0 \) is the unknown initial attitude quaternion at time \( t_0 \), \( \omega_0 \) is the unknown 3x1 initial angular velocity vector at time \( t_0 \) in body axes, and

\[
F(\omega_0) = 2 \begin{bmatrix}
\frac{1}{\sigma_1} q_{c1}^T \Phi(t_1, t_0; \omega_0) \\
\frac{1}{\sigma_1} q_{d1}^T \Phi(t_1, t_0; \omega_0) \\
\frac{1}{\sigma_2} q_{c2}^T \Phi(t_2, t_0; \omega_0) \\
\frac{1}{\sigma_2} q_{d2}^T \Phi(t_2, t_0; \omega_0) \\
\vdots \\
\frac{1}{\sigma_m} q_{cm}^T \Phi(t_m, t_0; \omega_0) \\
\frac{1}{\sigma_m} q_{dm}^T \Phi(t_m, t_0; \omega_0)
\end{bmatrix}
\]

with \( \Phi(t_i, t_0; \omega) \) being the orthogonal transition matrix for quaternions that satisfies \( q(t_0) = \Phi(t_i, t_0; \omega)q_0 \) for \( i = 1, \ldots, m \).

The initial value problem in Eqs. (9a) and (9b) can be integrated analytically using simple trigonometric functions in some cases. A general analytic solution exists, but it is expressed in terms of highly specialized functions. The complexity of this latter solution and the desire to develop a generalized algorithm imply that a numerical solution may be more practical. Therefore, the remainder of this paper will assume that a numerical solution will be used, one based on Runge-Kutta numerical integration.

Also needed in this paper's later developments are first and second partial derivatives of the matrix \( F(\omega_0) \) taken with respect to the elements of \( \omega_0 \). The needed partial derivatives are computed by taking the corresponding derivatives of Eqs. (9a) and (9b) and interchanging the order of differentiation and partial differentiation with respect to elements of \( \omega_0 \) on these equations' left-hand sides. These operations yield differential equations for the partial derivatives of the elements of \( \omega_0(t) \) and \( \Phi(t, t_0; \omega_0) \). The initial values of these partial derivatives are deduced via partial differentiation with respect to the corresponding elements of \( \omega_0 \) of the initial conditions \( \Phi(0, 0; \omega_0) = I_4 \) and \( \Phi(t_0, t_0; \omega_0) = I_4 \). The resulting initial value problems for the partial derivatives can be numerically integrated in order to compute the needed partial derivatives of \( \Phi(t, t_0; \omega_0) \) at the times \( t_1, \ldots, t_m \).
III. Solution of the Generalized Wahba Problem using a Combined Analytic/Numeric Technique

A. Outer Numerical Solution for the Rate Vector Wrapped Around an Inner Analytical Solution for the Quaternion

An important property of the original Wahba problem and its q-method solution is the ability to calculate the global optimum analytically in terms of an eigenvalue solution, as in Section II.A. The alternative SVD solution introduced later in that section is essentially an equivalent technique. It would be desirable to solve the generalized Wahba problem for a spinning spacecraft in a similar manner. Unfortunately, no analytic solution has been found except for a very restricted case\textsuperscript{11,12}.

The present section develops a method that uses a partial analytic solution and a partial numeric solution. It is a method that has been suggested in a latter part of Ref. 11. This solution exploits the form of the cost function in Eq. (7b) in order to exactly optimize $q_0$ for a given $\omega_0$. Given a fixed value of $\omega_0$, this "inner" $q_0$ optimization can be performed by using the SVD $F(\omega_0) = U(\omega_0)[S(\omega_0)0]^T(\omega_0)$. $q_{\text{opt}}(\omega_0)$ is equal to the $4^{th}$ column of the $4\times4$ orthonormal matrix $V(\omega_0)$, and $J_0[q_{\text{opt}}(\omega_0), \omega_0] = \frac{1}{2}s_{44}^2(\omega_0)$. If $\omega_0$ had somehow been chosen so as to minimize $\frac{1}{2}s_{44}^2(\omega_0)$, then $q_{\text{opt}}(\omega_0)$ and $\omega_0$ would constitute the solution to the generalized Wahba problem in Eqs. (7a)-(7c).

The complete solution algorithm is developed by wrapping an outer numerical optimization algorithm around the inner analytical $q_0$ optimization. The basic algorithm works as follows:

1) Start with a guess of $\omega_0$.
2) Compute $\mathbf{q}(t_0;\omega_0)$ for $i = 1, ..., m$ and use these results to compute the matrix $F(\omega_0)$.
3) Compute $q_{\text{opt}}(\omega_0)$ and $J_i(\omega_0) \equiv J_0[q_{\text{opt}}(\omega_0), \omega_0] = \frac{1}{2}s_{44}^2(\omega_0)$ by using the SVD of $F(\omega_0)$.
4) Check whether $\omega_0$ optimizes $J_i(\omega_0)$. If so, then stop; otherwise, proceed to Step 5).
5) Compute an increment $\Delta \omega_0$ such that $J_i(\omega_0+\Delta \omega_0) < J_i(\omega_0)$.
6) Replace $\omega_0$ by $\omega_0+\Delta \omega_0$ and go to Step 2).

The key to developing a good outer numerical optimization algorithm for $\omega_0$ is to use a good algorithm for computing $\Delta \omega_0$ in Step 5). A good algorithm is one that induces the most rapid possible decrease in $J_i(\omega_0+\Delta \omega_0)$ as compared to $J_i(\omega_0)$, thereby causing the new solution guess $\omega_0+\Delta \omega_0$ to be as near as possible to the final optimal solution. The next sub-section describes the needed algorithm.

B. Computation of the Optimal Rate Increment

The increment to the initial angular rate, $\Delta \omega_0$, is computed using Newton's method with trust-region bounds. The calculation uses a quadratic approximation of the Lagrangian of the original cost function in Eq. (7b), and it calculates a corresponding increment to $q_{\text{opt}}$ in the process. Such an approach obviates the need to explicitly compute first and second partial derivatives of the function $J_i(\omega_0) = \frac{1}{2}s_{44}^2(\omega_0)$, thereby avoiding the possibility of encountering singularities where the derivative with respect to $\omega_0$ is discontinuous, as in Ref. 11.

The Lagrangian function adjoins the constraint in Eq. (7c) to the cost function in Eq. (7b) using the Lagrange multiplier value $\lambda = s_{44}^2$ from the SVD of $F(\omega_0)$.

$$L_0(q_0, \omega_0) = \frac{1}{2}F(\omega_0)q_0^T F(\omega_0)q_0 + \frac{\lambda}{2}(1-q_0^T q_0)$$ \hspace{1cm} (10)

Let the attitude-rate increment be expressed in terms of the non-dimensional increment $\Delta \omega_0$ such that $\Delta \omega_0 = \Delta \omega_0/\Delta \omega_{\text{max}}$, where $\Delta \omega_{\text{max}}$ is the maximum of $(t_i-t_0)$ maximized over all $i$ values from 1 to $(m-1)$. Suppose, also, that the quaternion increment is expressed in terms of the $3\times1$ multiplicative increment $\Delta q_0$ such that the new unit-normalized quaternion is $q_{\text{new}} = \{([\Delta q_0,1]/(1+\Delta q_0^T \Delta q_0)^{1/2}) \otimes q_{\text{opt}}(\omega_0)$). Note that this new quaternion is guaranteed to satisfy the quaternion normalization constraint in Eq. (7c) for any choice of $\Delta q_0$. Therefore, the normalization constraint can be dropped from the definition of the approximate optimization problem that will be used to determine $\Delta q_0$ and $\Delta \omega_0$.

Given the foregoing definitions of the attitude-rate and quaternion increments, the following problem determines the locally optimal values of these increments subject to a quadratic trust-region bound:
find: \( \Delta q_0 \) and \( \Delta \omega_0 \)

to minimize: \( \Delta L_c(\Delta q_0, \Delta \omega_0) = g_{\omega}^T \Delta \omega_0 + \frac{1}{2} [\Delta q_0^T, \Delta \omega_0^T] \begin{bmatrix} H_{qq} & H_{q\omega}^T \\ H_{\omega q} & H_{\omega\omega} \end{bmatrix} \begin{bmatrix} \Delta q_0 \\ \Delta \omega_0 \end{bmatrix} \)

subject to: \( \Delta q_0^T \Delta q_0 + \Delta \omega_0^T \Delta \omega_0 \leq \rho^2 \)

where the terms on the right-hand side of Eq. (11b) constitute the quadratic approximation of the Lagrangian in Eq. (10) about the point \( [q_{\text{opt}}(\omega), \omega] \) and where the quadratic inequality constraint in Eq. (11c) constitutes the trust-region bound.

The 3x1 vector \( g_{\omega} \) and the 3x3 matrices \( H_{qq}, H_{q\omega}, \) and \( H_{\omega\omega} \) in Eq. (11b) are computed based on first and second partial derivatives of the Lagrangian in Eq. (10). Their formulas involve factors that arise from the transformations from \( q_0 \) to \( \Delta q_0 \) and from \( \omega \) to \( \Delta \omega_0 \). They are

\[
g_{\omega} = \frac{1}{\Delta \tau_{\text{max}}} \begin{bmatrix} (Fq_{\text{opt}})^T \frac{\partial F}{\partial \omega_x} q_{\text{opt}} \\ (Fq_{\text{opt}})^T \frac{\partial F}{\partial \omega_y} q_{\text{opt}} \\ (Fq_{\text{opt}})^T \frac{\partial F}{\partial \omega_z} q_{\text{opt}} \end{bmatrix}
\]

\[
H_{qq} = Q_p \begin{bmatrix} \frac{\partial^2 F}{\partial \omega_x^2} q_{\text{opt}} & \frac{\partial^2 F}{\partial \omega_x \partial \omega_y} q_{\text{opt}} & \frac{\partial^2 F}{\partial \omega_x \partial \omega_z} q_{\text{opt}} \\ \frac{\partial^2 F}{\partial \omega_y \partial \omega_x} q_{\text{opt}} & \frac{\partial^2 F}{\partial \omega_y^2} q_{\text{opt}} & \frac{\partial^2 F}{\partial \omega_y \partial \omega_z} q_{\text{opt}} \\ \frac{\partial^2 F}{\partial \omega_z \partial \omega_x} q_{\text{opt}} & \frac{\partial^2 F}{\partial \omega_z \partial \omega_y} q_{\text{opt}} & \frac{\partial^2 F}{\partial \omega_z^2} q_{\text{opt}} \end{bmatrix} Q_p^{-1}
\]

\[
H_{q\omega} = \frac{1}{\Delta \tau_{\text{max}}} \begin{bmatrix} (Fq_{\text{opt}})^T \frac{\partial^2 F}{\partial \omega_x \partial \omega_0} q_{\text{opt}} & (Fq_{\text{opt}})^T \frac{\partial^2 F}{\partial \omega_x \partial \omega_y} q_{\text{opt}} & (Fq_{\text{opt}})^T \frac{\partial^2 F}{\partial \omega_x \partial \omega_z} q_{\text{opt}} \\ (Fq_{\text{opt}})^T \frac{\partial^2 F}{\partial \omega_y \partial \omega_0} q_{\text{opt}} & (Fq_{\text{opt}})^T \frac{\partial^2 F}{\partial \omega_y \partial \omega_y} q_{\text{opt}} & (Fq_{\text{opt}})^T \frac{\partial^2 F}{\partial \omega_y \partial \omega_z} q_{\text{opt}} \\ (Fq_{\text{opt}})^T \frac{\partial^2 F}{\partial \omega_z \partial \omega_0} q_{\text{opt}} & (Fq_{\text{opt}})^T \frac{\partial^2 F}{\partial \omega_z \partial \omega_y} q_{\text{opt}} & (Fq_{\text{opt}})^T \frac{\partial^2 F}{\partial \omega_z \partial \omega_z} q_{\text{opt}} \end{bmatrix} Q_p^{-1}
\]

In these formulas, \( Q_p \) is the 4x3 matrix that projects the effects of the 3x1 multiplicative error quaternion \( \Delta q_0 \) into the 4-dimensional quaternion space of \( q_0 \). This matrix is
\[
Q_p = \begin{bmatrix}
(q_{0opt})_4 & -(q_{0opt})_3 & (q_{0opt})_2 \\
(q_{0opt})_3 & (q_{0opt})_4 & -(q_{0opt})_1 \\
-(q_{0opt})_2 & (q_{0opt})_1 & (q_{0opt})_4 \\
-(q_{0opt})_1 & -(q_{0opt})_2 & -(q_{0opt})_3
\end{bmatrix}
\]

with \((q_{0opt})_j\) indicating the \(j\)th element of \(q_{0opt}\). Note that \(F\) and \(q_{0opt}\) in Eqs. (12a)-(13) both depend on \(\omega\). This dependence is not explicitly shown in the interest of keeping these long formulas from becoming even longer.

Note that the choices of the Lagrange multiplier \(\lambda\) and of the projection matrix \(Q_p\) combine to ensure that \(H_{eq}\) in Eq. (12c) is positive definite if \(s_{33} > s_{44}\), i.e., if the minimum singular value of \(F(\omega)\) is unique. If there are two or more equal minimum singular values, then \(H_{eq}\) is only positive semi-definite. Nothing can be said about the matrix \(H_{eq}\) at a general \(\omega\) point, but it must be at least positive semi-definite at the optimal \(\omega\), and will usually be positive definite.

The trust-region radius \(\rho\) and the trust-region inequality in Eq. (11c) define a region in \([\Delta q_0; \Delta \omega_0]\) space. The trust-region optimization approach adaptively updates \(\rho\) in order to ensure that the quadratic expansion of the Lagrangian in Eq. (11b) is a reasonably good approximation of the cost in Eq. (7b) within the \(\rho\) trust region \(^{14,16}\). The actual method of initializing and adapting \(\rho\) is addressed in the next sub-section.

The solution to the step-size problem in Eqs. (11a)-(11c) normally takes the form:

\[
\begin{bmatrix}
\Delta q_0 \\
\Delta \omega_0
\end{bmatrix}_{opt} = -\begin{bmatrix}
H_{qq} & H_{q\omega}^T \\
H_{q\omega} & H_{\omega\omega}
\end{bmatrix}^{-1} \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

where \(\gamma\) is the Kuhn-Tucker multiplier associated with the trust-region inequality constraint in Eq. (11c) \(^{16}\). This multiplier is always non-negative and greater than or equal to the negative of the minimum eigenvalue of the \(6x6\) projected Hessian matrix that appears in the brackets on the right-hand side of Eq. (14). Therefore, the matrix that is inverted on the right-hand side of Eq. (14) is at least positive semi-definite, and it is usually positive definite.

In cases where the Hessian matrix is positive definite, the solution algorithm first tries the Kuhn-Tucker multiplier value \(\gamma = 0\) in Eq. (14). Next, it checks the resulting \([\Delta q_0; \Delta \omega_0]\)_{opt} to find out whether it respects the trust-region bound in Eq. (11c). If it does, then this is the solution. If the solution does not satisfy Eq. (11c), then the Eq. (14) formula for \([\Delta q_0; \Delta \omega_0]\)_{opt} is substituted into Eq. (11c), and the inequality is assumed to be satisfied as an exact equality. This results in a scalar equation for \(\gamma\). This equation can be transformed into an \(8^{th}\)-order polynomial, which can be solved using numerical techniques. The correct \(\gamma\) is the most positive real solution. Note that the transformation of this equation into an \(8^{th}\)-order polynomial is facilitated by first computing the eigenvalue decomposition of the Hessian matrix in Eq. (14).

If the Hessian matrix is indefinite or only positive semi-definite, then there exists a non-negative minimum value of \(\gamma\). Let this be \(\gamma_{meig}\). As noted above, \(\gamma_{meig}\) will equal the negative of the minimum eigenvalue of the Hessian matrix. Let the eigenvector associated with this eigenvalue be \([\Delta q_{meig}; \Delta \omega_{meig}]\). The solution procedure for the problem in Eqs. (11a)-(11c) proceeds by substituting Eq. (14) into Eq. (11c), re-arranging it to yield an \(8^{th}\)-order polynomial in \(\gamma\) solving the polynomial, and selecting the largest real-valued solution. If this solution is larger than \(\gamma_{meig}\), then it is used in Eq. (14) in order to compute the correct \([\Delta q_0; \Delta \omega_0]\)_{opt}.

The case where the maximum \(\gamma\) solution equals \(\gamma_{meig}\) is a special case. In this situation, the gradient vector \([0; \omega]\) has zero projection along the minimum eigenvector \([\Delta q_{meig}; \Delta \omega_{meig}]\) and the equation

\[
\begin{bmatrix}
H_{qq} & H_{q\omega}^T \\
H_{q\omega} & H_{\omega\omega}
\end{bmatrix} + \gamma_{meig} \begin{bmatrix}
0 \\
\omega
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

has a solution even though the \(6x6\) coefficient matrix on its left-hand side is singular. In fact, it has multiple solutions. The minimum-norm solution to this equation can be calculated by using the same eigenvalue decomposition of the Hessian that has been used to develop the \(\gamma\) polynomial. Suppose that this minimum-norm solution is \([\Delta q_{0\text{norm}}; \Delta \omega_{0\text{norm}}]\). This solution will obey the inequality in Eq. (11c), probably as a strict inequality.

The final optimal solution to the problem is then computed by forming:
That is, it is a linear combination of the minimum-norm solution to Eq. (15) and the minimum eigenvector of the Hessian matrix. The scalar factor \( \alpha \) is chosen by substituting the optimal solution in Eq. (16) into the constraint in Eq. (11c), treating the constraint as an exact equality, and solving it for \( \alpha \). It will be a quadratic equation in \( \alpha \) with real positive and negative solutions of equal magnitude. Either solution will suffice in Eq. (16).

A 2-dimensional analog of this situation is illustrated in Fig. 1. The black circle represents the trust-region boundary corresponding to Eq. (11c), and the dashed curves represent contours of the quadratic approximation of the Lagrangian in Eq. (11b). The grey point near the center of the figure represents the minimum-norm solution to the optimality necessary condition in Eq. (15), \( \Delta q_0 \). Note how it lies interior to the trust-region boundary. The two black points on the trust-region boundary represent the two optimal points from Eq. (16), each with a different sign of \( \alpha \). It is obvious from the contours that these are the two minimum-cost points that respect the trust-region bounds.

The techniques used here are similar to those used in Section II.C of Ref. 6. That section of that reference solves a similar quadratically constrained quadratic program.

**Fig. 1.** Trust-region solution of an example 2-dimensional quadratic problem with an indefinite Hessian.

### C. Trust-Region Algorithm for Numerical Solution of Outer \( \omega _0 \) Optimal Estimation Problem

The outer \( \omega _0 \) numerical solution algorithm uses trust-region techniques in order to implement the general algorithm of Section III.A in a way that ensures a cost decrease in its 5th step. The implemented algorithm is a slight adaptation of one from Ref. 16. It takes the form:

a) Start with a guess of \( \omega _0 \) and set \( \rho = \pi /4 \).
b) Compute \( \Phi (t_i, t_0, \omega _0) \) and its first and second partial derivatives with respect to \( \omega _0 \) for \( i = 1, \ldots , m \) and use these results to compute the matrix \( F(\omega _0) \) along with its first and second partial derivatives with respect to \( \omega _0 \). Initialize the counter \( j_\rho = 0 \).
c) Compute \( q_{opt}^{(i)}(a_i) \) and \( J_c(a_i) \equiv \int q_{opt}^{(i)}(a_i) \) by using the SVD of \( F(a_i) \). Also compute the vector \( g_{a_i} \) and the matrices \( H_{a_i}, H_{spa}, \) and \( H_{a_st} \).

d) Check whether \( a_i \) optimizes \( J_c(a_i) \). If so, then stop; otherwise, proceed to Step e).

e) Compute the angular rate increment \( \Delta \omega = \bar{\omega}_{opt} / \Delta \bar{\omega}_{max} \) after solving the problem in Eqs. (11a)-(11c) for \( \Delta q_{opt} \) and \( \Delta \bar{\omega}_{opt} \).

f) Compute \( \Phi_{f_i} (a_i + \Delta a_i) \) for \( i = 1, \ldots, m \), use these results to compute the matrix \( F(a_i + \Delta a_i) \), and use the SVD of \( F(a_i + \Delta a_i) \) to compute \( J_c(a_i + \Delta a_i) = \frac{1}{2} s_{J_c}^2 (a_i + \Delta a_i) \).

g) Compute the ratio \( \eta = [J_c(a_i + \Delta a_i) - J_c(a_i)] / \Delta L_c q_{opt} \Delta \bar{\omega}_{opt} \) and compute \( \rho_{actual} = (\Delta q_{opt} q_{opt}^T + \bar{\omega}_{opt} \bar{\omega}_{opt}^T) ^{1/2} \).

h) If \( 0.75 \leq \eta < 1.25 \) and \( \rho_{actual} = \rho \), then replace \( \rho \) by \( \min(2, \rho_{actual}, \pi / 4) \) and replace \( J_c \) by \( J_{c + 1} \).

i) If \( \eta \leq 0 \) and \( j_p = 30 \), then go to Step e). If \( \eta \leq 0 \) and \( j_p > 30 \), then stop unsuccessfully. Otherwise, proceed to Step j).

j) Replace \( a_i \) by \( a_i + \Delta a_i \) and go to Step b).

This algorithm starts with the trust-region diameter \( \rho = \pi / 4 \), and this diameter gets adapted in Step h), either up or down, depending on the ratio \( \eta \). The ratio \( \eta \) is a measure of how well the quadratic Lagrangian approximation in Eq. (11b) models the actual changes of the cost function \( J_c(a_i) \). If it models them well, then \( \eta \) will be near 1, and it is reasonable to increase the trust-region size if the \( \Delta q_{opt} \) solution was on the trust-region boundary. If, on the other hand, \( \eta \) is too small, then the Lagrangian does not model the changes in \( J_c(a_i) \) well, and the trust region must be decreased. If the ratio \( \eta \) is non-positive, then \( J_c(a_i + \Delta a_i) \geq J_c(a_i) \) because \( \Delta L_c (\Delta q_{opt} \Delta \bar{\omega}_{opt}) < 0 \) is guaranteed. In this case, Step i) rejects \( \Delta a_i \) and returns to Step e) in order to calculate a new \( \Delta a_i \) based on a new, smaller trust-region bound \( \rho \), one that is determined in Step h).

Step h) ensures that the trust-region bound never goes above the \( \pi / 4 \) limit. This limit of an eighth of a cycle is a heuristic choice that seeks to avoid potential aliasing problems. The imposition of this limit is one of the two modifications of this algorithm from that of Ref. 16. The other modification is to not increase \( \rho \) if \( \eta \geq 1.25 \).

Various algorithm termination criteria are used in Step d) to check whether \( a_i \) optimizes \( J_c(a_i) \). The first test asks whether the Hessian of the Lagrangian approximation in Eq. (11b) is positive definite. A related second test asks whether the most recent \( \eta \) ratio obeys \( 0.97 \leq \eta \leq 1.03 \). Termination cannot occur if these two tests have not been passed.

A third criterion tests whether \( J_c(a_i) \) is already very small. Two reasonable measures of the expected size of \( J_c(a_i) \) are \( [(1/2) + (1/2) + \ldots + (1/2)] \), which is the first term on the right-hand side of Eq. (2b), and \( (2m-6)/2 \), which is based on the reasonable statistical model that \( 2L_c(q_{opt}) \) should be a sample from a chi-squared distribution of degree equal to the effective number of scalar attitude measurements, 2 per unit direction vector measurement, minus the number of estimated scalar quantities, 3 free initial attitude parameters and 3 initial attitude rates. If \( J_c(a_i) \) divided by the largest of these two candidate values is less than 2.22x10^-17, then the optimization is terminated. This criterion normally does not lead to termination except in cases where \( m = 3 \), i.e., where the optimization amounts to equation solving so that the optimal \( J_c(a_i) \) is zero.

Two additional termination tests check whether \( \eta \) and whether \( |\Delta a_{opt}| << 1 + J_c(a_i) \). If \( |\Delta a_{opt}| \leq 10^{-15}(1+J_{max}(a_i)) \) OR if \( M_c(\Delta q_{opt}, \Delta \bar{\omega}_{opt}) < 10^{-20}[1+J_c(a_i)] \), then the algorithm terminates. An alternative termination condition occurs if \( |\Delta a_{opt}| \leq 10^{-19}(1+J_{max}(a_i)) \) AND \( M_c(\Delta q_{opt}, \Delta \bar{\omega}_{opt}) < 10^{-17}[1+J_c(a_i)] \). The small values of the coefficients used in these termination criteria are heuristically chosen to be on the order of machine precision for double-precision arithmetic. Therefore, they ensure that the solution converges nearly to within machine precision of the true minimum. Of course, none of these conditions produces termination if the Hessian of the Lagrangian is not positive definite or if \( \eta \) is outside the range \( 0.97 \leq \eta \leq 1.03 \).

A final termination condition occurs if too many decreases of \( \rho \) occur in Step h) without achieving \( \eta > 0 \) in Step i). This condition is implemented by checking whether \( j_p > 30 \) in Step i). This normally occurs when the algorithm is essentially at the minimum, but its other termination criteria have failed to stop its execution. Tests for the positive definiteness of the Lagrangian Hessian and for the size of the cost gradient \( g_a \) tend to confirm that the algorithm has actually converged.
The trust-region method is able to deal with an indefinite Hessian far from the solution and still converge. This ability is embedded in the calculation of \( \Delta \omega \) via solution of the quadratically constrained quadratic program in Eqs. (11a)-(11c). This ability enables the method to successfully negotiate local maxima and saddle points in \( J_c(\omega) \) on its way to finding a minimum. Less obvious, however, is the method's ability to deal with discontinuities in the first derivative of \( J_c(\omega) \). Such discontinuities occur when the inner Wahba problem has more than one \( q_{\text{opt}} \), which occurs when the matrix \( F(\omega) \) has two equal minimum singular values, i.e., when \( s_{33}(\omega) = s_{44}(\omega) \). In this case, \( H_{qq} \) in Eq. (11b) will have a zero eigenvalue and be positive semi-definite, and the entire Hessian of the Lagrangian in Eq. (11b) will normally be indefinite. The trust region's ability to handle an indefinite Hessian enables it to decide among multiple possible descent directions in order to choose a \( \Delta \omega \) search step that produces a decrease of \( J_c(\omega + \Delta \omega) \) relative to \( J_c(\omega) \). Thus, it solves the problem of how to search from a \( J_c(\omega) \) performance metric cusp, as depicted in Fig. 1 of Ref. 11.

D. Finding the Global Minimum

It is possible for \( J_c(\omega) \) to have multiple local minima, as demonstrated in Refs. 11 and 12. The trust-region algorithm of the previous section is only guaranteed to find a local minimum of the cost function \( J_c(\omega) \). Therefore, extra measures must be implemented in order to guarantee that the global minimum is found.

A reasonable approach to finding the global minimum is to re-start the trust-region algorithm from the various elements of a set of \( \omega \) initial guesses. This set should be chosen to span the possible range of true \( \omega \) values and to have sufficient density to ensure that at least one of the first guesses lies within the trust-region algorithm's convergence domain for the global minimum.

The set of possible values of \( \omega \) can be defined in terms of aliasing limits. As in Ref. 12, reasonable limits are that the body-axis and inertial nutation frequencies be less than \( \pi / \Delta t_{\text{max}} \). These limits translate into the following constraints that can be defined in terms of the body-axis angular momentum vector \( h_{sc} = I_{sc} \omega \),

\[
| e_s^T h_{sc} | \leq I_s \left( \frac{l_{11} l_{22}}{(l_{11} - l_s)(l_{22} - l_s)} \right) \frac{\pi}{\Delta t_{\text{max}}} \equiv h_{s_{\text{max}}} \tag{17a}
\]

\[
|h_{sc}| \leq \min(l_{11}, l_{22}) \left( \frac{\pi}{\Delta t_{\text{max}}} \right) \equiv h_{\text{max}} \tag{17b}
\]

where \( e_s \) is the unit direction vector in body axes of the nominal spin axis and where \( l_{11}, l_{22}, \text{ and } l_s \) are the three principal moments of inertia, that is the three eigenvalues of \( I_{sc} \). \( l_s \) is the inertia about the nominal spin axis, and \( l_{11} \) and \( l_{22} \) are the inertias about the two transverse axes. Note that \( e_s \) is an eigenvector of \( I_{sc} \) corresponding to the eigenvalue \( l_s \). The two inertias \( l_{11} \) and \( l_{22} \) are assumed to be either both less than \( l_s \), for a major-axis spinner, or both greater than \( l_s \), for a minor-axis spinner. The limit in Eq. (17a) enforces an upper bound on the body-axis nutation rate, and the limit in Eq. (17b) enforces an upper bound on the inertial nutation rate. An additional upper bound on the magnitude of the spin rate also makes sense. It takes the form:

\[
| \omega | \leq \left( \frac{\pi}{\Delta t_{\text{max}}} \right) \equiv \omega_{\text{max}} 
\tag{18}
\]

Given the computed limits \( h_{s_{\text{max}}}, h_{\text{max}}, \) and \( \omega_{\text{max}} \), the following algorithm samples an acceptable random first guess of \( \omega \) from a distribution that is reasonably uniform over the region defined by the inequalities in Eqs. (17a), (17b), and (18):

i. Sample \( h_{\text{raw}} \) from a 3-dimensional vector Gaussian distribution with a mean of 0 and a covariance equal to the identity matrix.

ii. Compute the angular momentum unit direction vector \( \hat{h}_{sc} = h_{\text{raw}} / ||h_{\text{raw}}|| \).

iii. Sample \( h_{\text{mean}} \) from the uniform distribution between 0 and 1.

iv. Compute \( h_{\tilde{sc}} = \hat{h}_{sc} h_{\text{max}}(h_{\text{mean}})^{1/3} \).

v. If \( | e_s^T h_{sc} | > h_{s_{\text{max}}} \) then return to Step i; otherwise proceed to Step vi.

vi. Compute \( \omega = I_{sc}^{-1} h_{sc} \).

vii. If \( || \omega || > \omega_{\text{max}} \), then return to Step i; otherwise accept \( \omega \) and terminate successfully.

The overall batch estimation algorithm is a concatenation of this algorithm for generating first guesses and the trust-region algorithm of Section III.C. The overall algorithm proceeds as follows: The user selects the number of first guesses that will be used to implement the global search, \( N \). Next, the algorithm executes this sub-section's
first-guess selection algorithm $N$ times in order to generate $N$ independent candidate first guesses of $\mathbf{a}_b$. Let these be designated $\mathbf{a}_{b1}$, $\mathbf{a}_{b2}$, $\mathbf{a}_{b3}$, ..., $\mathbf{a}_{bN}$. Next, the algorithm carries out the trust-region optimization of Section III.C $N$ times starting from these first guesses in order to generate the $N$ optimal solutions $\mathbf{a}_{b1}$, $\mathbf{a}_{b2}$, $\mathbf{a}_{b3}$, ..., $\mathbf{a}_{bN}$ along with their optimal costs $J_c(\mathbf{a}_{b1})$, $J_c(\mathbf{a}_{b2})$, $J_c(\mathbf{a}_{b3})$, ..., $J_c(\mathbf{a}_{bN})$ and their corresponding optimal initial attitude quaternions $\mathbf{q}_{b1}$, $\mathbf{q}_{b2}$, $\mathbf{q}_{b3}$, ..., $\mathbf{q}_{bN}$. Finally, the algorithm selects as the optimal estimate the one that has the lowest $J_c$ cost value. Thus, if $J_c(\mathbf{a}_{b_k}) \leq J_c(\mathbf{a}_{b_j})$ for all $k = 1$, ..., $N$, then the optimal estimates of the initial attitude and attitude rate are, respectively, $\mathbf{q}_{opt} = \mathbf{q}_{b_k}$ and $\mathbf{a}_{opt} = \mathbf{a}_{b_k}$.

Note that the algorithm should also verify that $J_c(\mathbf{a}_{b_k})$ is on the order of 0.5(2m-6) or less, as expected from the chi-square model of 2$J_c(\mathbf{a}_{b_k})$. If the optimal cost is significantly greater than 0.5(2m-6), then the global minimum may not yet have been found. If such is the case, then it would be wise to increase the required additional random first guesses of $\mathbf{a}_b$, and run the trust-region algorithm from each new first guess.

If this strategy fails to produce a $J_c(\mathbf{a}_{b_k})$ on the order of 0.5(2m-6), then the problem model in Section II may be in error. Possible model errors are incorrect measurement error standard deviations, incorrect sensor alignment, or an incorrect rigid-body dynamics model. Problems with the rigid-body model could be caused by an incorrect moment-of-inertia matrix $I_\omega$, the presence of significant external torques, or non-rigid-body effects such as fuel slosh or the vibration of flexible structural members.

E. Estimation Error Covariance

The estimation error covariance matrix takes the form:

$$P_0 = E\left[ \Delta \mathbf{q}_{\text{err}} \left( \mathbf{a}_0 \right) \Delta \mathbf{q}_{\text{err}}^T \left( \mathbf{a}_0 \right) \right] = \left[ \begin{array}{cc} H_{qq} & A_{t_{\text{max}}} H_{q\theta}^T \\ A_{t_{\text{max}}}^T H_{q\theta} & A_{t_{\text{max}}}^2 H_{\theta\theta} \end{array} \right]^{-1} \tag{19}$$

where $\Delta \mathbf{q}_{\text{err}}$ constitutes the first three components of the multiplicative error quaternion:

$$\Delta \mathbf{q}_{\text{err}} = \mathbf{Q}_p^T \mathbf{q}_0 \tag{20}$$

with the 3x4 matrix $\mathbf{Q}_p^T$ being the transpose of the matrix defined in Eq. (13) evaluated at the optimal estimate $\mathbf{q}_{opt}$.

Recall that the elements of $\Delta \mathbf{q}_{\text{err}}$ are related to the optimal estimate $\mathbf{q}_{opt}$ and the true quaternion $\mathbf{q}_0$ by the quaternion multiplication equation:

$$\mathbf{q}_0 = \frac{1}{\sqrt{1 + \Delta \mathbf{q}_{\text{err}}^T \Delta \mathbf{q}_{\text{err}}}} \left[ \begin{array}{c} \Delta \mathbf{q}_{\text{err}} \\ 1 \end{array} \right] \otimes \mathbf{q}_{opt} \tag{21}$$

IV. Truth-Model Simulation

A truth-model simulation has been used to generate data in order to test the new algorithm. The truth-model simulation starts with truth values of $\mathbf{q}_0$ and $\mathbf{a}_0$. It uses $\mathbf{a}(t_0) = \mathbf{a}_0$ and $\mathbf{a}(t_0, t_0 ; \mathbf{a}_0) = I_\omega$, and it integrates the dynamics model in Eqs. (9a) and (9b) forward to times $t_1$, ..., $t_m$, in order to compute $\mathbf{a}(t_i, t_0 ; \mathbf{a}_0)$ for $i = 1$, ..., $m$.

Given truth values of the inertial unit-direction reference vectors $\mathbf{r}_i$ for $i = 1$, ..., $m$, the simulated measurements of the corresponding body-axis unit-direction vectors are generated using the equation:

$$b_i = \frac{A(q_i) r_i + \sigma \nu_i}{\sqrt{[A(q_i) r_i + \sigma \nu_i]^T [A(q_i) r_i + \sigma \nu_i]}} \quad \text{for} \quad i = 1, ..., m \tag{22}$$

where $\mathbf{\nu}$ is sampled from a 3x1 vector Gaussian distribution with a mean of zero and a covariance equal to the identity matrix. The unit-normalized noise vectors $\mathbf{\nu}$ and $\mathbf{\nu}$ are sampled in a way that leaves them uncorrelated for $i \neq j$.

Given that this is mostly a theory paper, only a small set of truth-model simulations have been run. One has been for a minor-axis spinner with principal inertias equal to 201, 191, and 21 kg-m$^2$. Another has been for a major-axis spinner with principal inertias 137, 149, and 217.1 kg-m$^2$. The chosen value for $\mathbf{a}_0$ for the minor-axis spinner causes the spin period to oscillate between 58.4 and 60.8 seconds due to nutation, with a body-axis nutation period of 74.2 sec and an inertial nutation period of 133 sec. The major-axis spinner has been simulated to rotate with a spin period of 77.1 sec and corresponding nutation periods of 150.6 sec in the body axes and 50.3 sec with
respect to inertial axes. The average nutation coning half-angles are 77.5 deg for the minor-axis spinner and 5.3 deg for the major-axis spinner.

Five vector measurements have been simulated for the minor-axis spinner spaced equally at 15 sec intervals. In all cases considered in this paper, \( t_0 \) is chosen to equal \( t_i \), which is allowable under this paper's assumptions. Thus, \( t_0 = t_1 = 0 \) sec, \( t_2 = 15 \) sec, \( t_3 = 30 \) sec, \( t_4 = 45 \) sec, and \( t_5 = 60 \) sec for the minor-axis case. Varying measurement accuracies have been assumed, ranging from a minimum \( \sigma_i \) of 0.93 deg to a maximum \( \sigma_i \) of 1.22 deg. The inertial reference vectors for this case, \( r_i \) for \( i = 1, ..., 5 \), have been generated randomly with no two being the same.

The major-axis spinner simulation involves 10 measurements spaced equally once every 10 seconds. That is, \( t_0 = t_1 = 0 \) sec, \( t_2 = 10 \) sec, \( t_3 = 20 \) sec, ..., \( t_{10} = 90 \) sec. The inertial reference vectors \( r_i \) for \( i = 1, ..., 10 \) alternate between two values for this case, with the two direction vectors 111.9 deg apart. Similarly, the corresponding \( \sigma_i \) values oscillate back and forth between 0.05 deg and 0.13 deg with one value associated with one reference vector and the other value associated with the other vector.

Two additional sets of cases have been considered. One set corresponds to the axially-symmetric major-axis spinner cases of Ref. 12 with various levels of sensor error. The other set corresponds to the minor-axis spinner of that same reference. These cases provide points of comparison with the equation-solving approach of Ref. 12. Contrary to the cases in that reference, however, each of these cases uses 5 vector measurements rather than the same reference. These cases provide points of comparison with the equation-solving approach of Ref. 12. This extension to more measurements is easy for the present technique and, as will be shown, demonstrates the improved ability of the present method to find a distinct globally optimal solution that provides good estimates of the truth attitude and rate.

V. Results of Truth-Model Simulation Study

The new solution algorithm for the generalized Wahba problem has worked well for all of the truth-model simulation cases. This paper's new minor-axis spinner case has been solved starting from \( N = 20 \) randomly generated guesses of \( \omega_0 \). Only 17 of these cases converged to a sensible local minimum within 30 trust-region iterations to find 14 distinct local minima. The global minimum was found, but starting only from one of 20 initial guesses of \( \omega_0 \). This global minimum, however, was obviously the correct solution.

The major-axis spinner case has used \( N = 1000 \) random initial \( \omega_0 \) guesses, 899 of which converged within 30 iterations. These 899 solutions produced a set of 25 distinct local minima. The global minimum was reached from 148 of these \( \omega_0 \) initial guesses. Given the 14.8 percent rate of convergence to the global minimum, it seems obvious that \( N = 20 \) initial guesses of \( \omega_0 \) would have sufficed to find it. The larger number \( N = 1000 \) has been used more as means of studying the convergence issues than as a practical value to use operationally.

The global and local minimum costs and their corresponding chi-squared probabilities are shown in Fig. 2 for these two cases. The upper plot is for the minor-axis spinner case, and the lower plot is for the major-axis spinner. The horizontal axis plots the locally minimizing cost \( J_c \) at each locally optimal solution \( \omega_{0opt} \). The vertical axis plots the corresponding chi-squared cumulative distribution function evaluated at \( 2J_c \). The degree of the chi-squared distribution used for the upper plot is \((2m-6) = 4\) because \( m = 5 \) vector measurements are available for this case, and the lower plot is for a degree-14 distribution because \( m = 10 \) for the major-axis spinner case. The important point of these two curves is that the two grey dots that represent the global minima of the two cases lie significantly to the left, i.e., at significantly lower costs, than the non-global minimum costs that are depicted by the diamonds. The reasonableness of these global solutions and the significance of their isolation on the low-cost left ends of these curves are emphasized by the fact that they both lie at points where the cumulative distribution is significantly below 1. This fact confirms that these global minimum cost values are reasonable values based on normal statistical measurement noise variations. The fact that all other local minima lie up near a cumulative probability of 1, on the other hand, indicates their unreasonableness.

A second test has been applied in order to check the reasonableness of the global solutions and of their corresponding covariance matrices. The test calculates the statistic

\[
\zeta = \left[ (Q_p^T q_{0true})^T, (\omega_{0true} - \omega_{0opt})^T \right] P_0^{-1} \left[ (Q_p^T q_{0true}) \right] \left( \omega_{0true} - \omega_{0opt} \right)
\]

where the 4x3 matrix \( Q_p \) is evaluated at \( q_{0opt} \) as in Eq. (13), except that \( q_{0opt} \) is first subjected to a sign change, if necessary, in order ensure that \( q_{0true}^T q_{0opt} \geq 0 \). Such a sign change does not affect the actual attitude representation, but it is needed in order for the \( \zeta \) statistic in Eq. (23) to be a sample from a chi-squared distribution
of degree 6. This statistic evaluates to $\zeta = 4.18$ for the minor-axis spinner case and to $\zeta = 7.76$ for the major-axis case. Both of these are reasonable samples from a degree-6 distribution.

![Fig. 2. Chi-squared probabilities of global and local solutions to two generalized Wahba problems, one for a minor-axis spinner (top plot) and one for a major-axis spinner (bottom plot).](image)

The actual total initial attitude error in $q_{opt}$ for the minor-axis case is 1.97 deg, and the expected value of this total error is 2.91 based on the trace of the 3x3 upper-left-hand block of the $P_0$ covariance matrix. Similarly, the major-axis case has a total initial attitude error 0.11 deg and an expected RMS value of 0.09 deg. It makes sense that the major-axis spinner case has this increased accuracy because it has twice as many measurements, and each of its measurements is significantly more accurate than each of the measurements for the minor-axis case.

The total initial attitude rate error in $\omega_{opt}$ is similarly small for both cases. The value of $||\omega_{opt} - \omega_{true}||$ is 0.165 deg/sec for the minor-axis spinner, which equals 2.7% of $||\omega_{true}||$. The corresponding expected value of this error based on $P_0$ is 0.140 deg/sec. For the major-axis spinner case, $||\omega_{opt} - \omega_{true}||$ is 0.00610 deg/sec, or 0.13% of $||\omega_{true}||$, with the corresponding expected level of this error being 0.00482 deg/sec. The improved attitude rate accuracy of the major-axis spinner case, both in terms of absolute accuracy and relative accuracy, is likely the result of the superior accuracy of its vector attitude measurements and the greater duration of its data batch: $t_m - t_1 = 90$ sec for this case versus 60 sec for the minor-axis case.

Additional evidence of the efficacy of the new algorithm has been provided by tests involving the axially-symmetric major-axis and minor-axis spinner cases of Ref. 12. In each case, this paper's new algorithm has successfully found a unique global optimum that provides a good estimate of the truth attitude and rate, an estimate whose error produces a reasonable Eq.-23 test statistic. Each of these cases has used only $N = 20$ randomized initial guesses of $\omega$, and each has produced a distinct global minimum that has a significantly lower cost than all other local minima.

These latter results contrast significantly with the ambiguity of the estimates obtained in some instances in Ref. 12 when using only 3 vector measurements. Recall that $m = 3$ is the minimum number of measurements required for local observability. As suggested by Ref. 12, the use of 5 measurements in the present case provides the needed information to distinguish the correct global minimum from local minima that provide poor estimates. The present algorithm provides the necessary framework for using the extra 2 vector measurements, or any number of extra measurements, in an optimal manner. Ref. 12 also uses extra measurements to remove ambiguities, but only in an ad hoc manner.
VI. Summary and Conclusions

This paper has developed and tested a global solution procedure for a generalized Wahba problem. The generalized problem determines the initial 3-axis attitude quaternion and attitude rate of a spinning spacecraft based on a time series of 3 or more attitude vector observations. The solution procedure is based on an inner-analytic solution for the attitude quaternion and an outer numerical search for the attitude rate. Given a guess of the initial attitude rate vector and the torque-free rigid-body dynamics model of the spacecraft, the inner quaternion problem reduces to a standard Wahba problem. The inner solution is based on a transformed version of the Wahba problem that can be solved using a singular value decomposition. The outer numerical solution for the attitude rate uses a trust-region implementation of Newton's method. The trust-region step-size calculation minimizes a quadratic approximation of a Lagrangian function with respect to multiplicative attitude-quaternion perturbations and additive attitude-rate perturbations. These perturbations are calculated subject to a norm-squared constraint on their magnitude. The outer trust-region algorithm adjusts the constraint magnitude in order to ensure global convergence to a local minimum. Solution for the global minimum of the generalized Wahba problem is achieved by re-starting the trust-region algorithm from a number of random initial attitude-rate guesses that lie within a region constrained by aliasing considerations.

The algorithm has been implemented and tested using a truth-model simulation. The trust-region outer solution tends to find each local minimum in significantly fewer than 30 major iterations. The global minimization strategy tends to find the true global minimum when starting from only 20 random guesses of the initial attitude rate. The inverse of the Lagrangian function's Hessian at the global optimum gives an estimation error covariance that is consistent with the modeled effects of the random attitude measurement errors. Per-axis 1-σ attitude accuracies on the order of 1 deg and better and per-axis rate errors on the order of 0.1 deg/sec and better have been demonstrated via truth-model simulation when using vector sensors with accuracies of 1 deg or better and batch intervals with lengths on the order of 60 sec or longer.

References