Solution Strategies for an Extension of Wahba’s Problem to Spinning Spacecraft

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Several methods have been devised to estimate the optimal attitude quaternion and angular velocity for an axially-symmetric spacecraft given only a time series of vector attitude measurements and a rigid-body model of the spacecraft dynamics. These methods solve a generalization of Wahba’s problem that is applicable to a low-complexity spinning spacecraft such as a sounding rocket, because it avoids the divergence issues of an EKF without employing expensive rate gyros. Measurements at different times are combined via the closed-form quaternion state transition matrix that makes use of Euler’s equations for a rigid body. Parameter transformations and algebraic manipulation result in partial solutions, and numerical methods can exploit known solution properties to obtain global solutions.

I. Introduction

Many spacecraft mission objectives, such as positioning of a solar panel or telescope, are strongly dependent on the spacecraft attitude relative to some reference frame. Consequently, mission success often depends on efficient and reliable methods of estimating attitude from available measurements. Even when accurate dynamics and measurement models are available, they are usually highly nonlinear. Nonlinearities can lead to failure of traditional estimation techniques, such as the Extended Kalman Filter (EKF), by divergence.1 Therefore, the principle objective of the present effort is to develop globally optimal solution techniques that cannot diverge.

Attitude estimation methods have often been formulated to solve what is known as “Wahba’s problem”:2 Given two or more vector measurements in body coordinates at a particular instant in time, corresponding to known inertial reference vectors, find the orthonormal rotation matrix (or equivalent unit quaternion) that minimizes the errors in a weighted-least-squares sense by minimizing the cost function:

\[ J(q) = \frac{1}{2} \sum_{i=1}^{m} \sigma_i^{-2} (b_i - A(q)r_i)^T [b_i - A(q)r_i] \]  

In this cost function, \( q \) is the unit-normalized quaternion that parameterizes the attitude, \( A(q) \) is the corresponding direction cosine matrix, \( b_i \) is the vector measurement in body axes, and \( r_i \) is the corresponding inertial reference vector. The measurements are weighted by the inverse square of the measurement error standard deviation \( \sigma_i \). Standard solutions to Wahba’s problem avoid errors due to linearization, which are inherent in an EKF. Consequently they are not subject to divergence and provide global optimality.

A major weakness of Wahba’s problem is that it assumes all measurements are available at a single instant in time, or that the spacecraft attitude is stationary. Thus, it does not adapt easily to a spacecraft undergoing dynamic motion when the measurements are separated by time. Many spacecraft mission objectives, however, are aided by using a spinning spacecraft. Such missions may limit the number of sensors in order to reduce costs. The elimination of rate gyros is one potential savings. Previous Wahba extensions that dealt with non-zero angular velocity have assumed the availability of rate gyros.3,4 Thus, the present work is a departure from those efforts.

A previous paper by one of the authors5 proposes two generalized versions of Wahba’s problem. They both seek to estimate the optimal initial quaternion and the spacecraft angular velocity given only a time series of vector attitude measurements. Reference 5 also presents a closed-form solution for a restricted version of one of these

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problems. Solutions to these proposed problems would enable both attitude and angular velocity estimation for a spinning spacecraft having only one vector sensor, such as a narrow-field-of-view star camera or a magnetometer. The divergence issues associated with an EKF would be avoided. A sequence of batch filters based on solutions to these problems could be used for non-divergent attitude and rate estimation, or one of these batch solutions could be used to initialize an EKF with a good state estimate that would help it to avoid divergence.

The present paper addresses a restricted form of the second problem proposed in Ref. 5. In this restricted form, an axially-symmetric, rigid spacecraft with negligible nutation damping undergoes torque-free motion. The inertia matrix is known, and is assumed diagonal without loss of generality. Single vector measurements are available at distinct times, and the initial attitude quaternion and initial angular velocity vector are sought. Additionally, the number of measurements is restricted to three, which causes the problem to be fully determined but not overdetermined. This problem is useful because (nearly) axially-symmetric, spinning spacecraft are common for simple applications, and the inertia matrix can usually be well characterized in advance. While the three-measurement case has limited applicability on its own, it is much more tractable than the case of an arbitrary number of measurements, and may even suggest some avenues towards the solution of this more difficult case. Furthermore, the three-measurement solution could be used to initialize an EKF to avoid divergence problems.

The present paper makes three contributions. First, it develops several specific sets of equations whose solution will solve the restricted problem. Second, it outlines a global numerical solution technique that can be applied to any such system of equations, along with several issues that may arise in numerical implementations. Third, it discusses the results of numerical solutions for three of the developed equation sets, as tested with simulated truth model data.

This paper is organized as follows: Section II summarizes some of the results of Ref. 5 and formulates the restricted problem in a way that closely resembles the form that yields the \( q \) method solution of Wahba’s problem. In Section III, the dynamics of the restricted problem are discussed, and the quaternion state transition matrix for the axially-symmetric case is presented by means of these dynamics. Section IV contains four partial analytical solutions, each of which reduce the problem to a set of three or four nonlinear equations in three or four unknowns. A global numerical approach is formulated in Section V, and some numerical issues are addressed. Section VI discusses the results of numerical implementation of the analytical solution approaches from Section IV. Section VII summarizes the paper’s results and draws its conclusions.

**II. Formulation of the Restricted Problem**

A well-known solution to Wahba’s problem is the \( q \) method. The \( q \) method rearranges the original cost function (Eq. (1)) to produce an equivalent optimal estimation problem:

\[
\begin{align*}
\text{find:} & \quad q \\
\text{to maximize:} & \quad p(q) = q^T K q \\
\text{subject to:} & \quad q^T q = 1
\end{align*}
\]

For the \( i \)th measurement, a symmetric 4x4 matrix \( K_i \) is formed from the \( b_i \) and \( r_i \) vectors. Then the symmetric 4x4 matrix \( K \) is formed from the individual \( K_i \) matrices as follows:

\[
K = \sum_{i=0}^{m-1} K_i
\]

After adjoining the unit-length constraint using the Lagrange multiplier \( \lambda \) and deriving the first-order optimality necessary condition, the \( q \)-method solution reduces to an eigenvalue problem. The optimal \( q \) is the eigenvector of \( K \) corresponding to its maximum eigenvalue, \( \lambda_{\text{opt}} \): \( K q_{\text{opt}} = \lambda_{\text{opt}} q_{\text{opt}} \).

The generalized Wahba’s problems posed in Ref. 5 extend this formulation to the case of a spinning spacecraft with only a time series of vector measurements available. The first form considers a spacecraft spinning around its known major principal inertia axis with enough nutation damping that the spin direction and rate can be assumed constant. In this problem, there are four unknowns: three to parameterize the initial attitude quaternion, and the unknown constant spin rate \( \omega \).
The second generalized problem, which this paper attempts to solve in a restricted form, builds on the first problem. Instead of assuming a constant, known spin axis, the second problem allows all three components of the angular velocity vector $\omega(t)$ to be unknown, but the moment-of-inertia matrix is assumed to be known and the resulting nutational motion is assumed to be torque-free. If the initial attitude quaternion $q_0$ and angular velocity $\omega_0$ were known, then one could simultaneously integrate Euler’s equation and the kinematic quaternion transition matrix differential equation forward in time in order to obtain the attitude and angular velocity at any time. Thus, six scalar unknowns fully parameterize the second problem: three associated with the initial quaternion and three associated with the initial angular velocity. A minimum of three vector measurements would be necessary to solve this problem. If the quaternion state transition matrix can be determined as a function of $\omega_0$, then the problem solution can be written as a $q$-method maximization:

find: $q_0$ and $\omega_0$

subject to:

$$ q_0^T q_0 = 1 $$

where $t_0$ is the initial time at which $q_0$ and $\omega_0$ apply, $t_i$ is the time of the $i^{th}$ measurement $b_i$ and the $i^{th}$ reference vector $r_i$, and $\Phi(t_i, t_0; \omega_0)$ is the quaternion transition matrix from time $t_0$ to time $t_i$ during attitude motions dictated by the value of $\omega_0$.

From the form of this maximization, it is clear that given $\omega_0$ and the equations for the quaternion state transition matrix $\Phi(t_i, t_0; \omega_0)$, the $K$ matrix could be computed and the optimal quaternion found by solving the corresponding $q$-method eigenvalue problem.

The problem form in Eqs. (4a) - (4c) suggests an inner-outer optimization procedure as discussed in Ref. 5. An inner optimization would be performed on $q_0$ to determine the optimal $q_0$ as a function of $\omega_0$, and an outer optimization would seek $\omega_0$ to maximize the maximum eigenvalue of the $K$ matrix. This outer optimization is made challenging by the fact that the maximum eigenvalue of a symmetric matrix depends on the matrix elements in a complicated way. Gradient-based search methods are likely to fail because the problem contains an unconstrained three-dimensional search space of possible angular velocities in which local maxima are likely, as are gradient discontinuities where the largest and second-largest eigenvalues exchange places.\(^3\)

### III. The Quaternion Transition Matrix for the Axially Symmetric Case

The restricted form of the second problem addressed in this paper considers an axially-symmetric rigid spacecraft undergoing torque-free motion with known inertia matrix $I_{sc}$ given by

$$ I_{sc} = \begin{bmatrix} I_T & 0 & 0 \\ 0 & I_T & 0 \\ 0 & 0 & I_S \end{bmatrix} $$

The nominal spin axis is the $z$ axis, and the two transverse axes have identical moments of inertia, consistent with the assumption of axial symmetry. Euler’s equation of motion for this system is given in many texts,\(^6\),\(^7\) and will not be repeated in full here. Several known results are important for this problem. The angular velocity in body axes can be written as

$$ \omega(t) = \begin{bmatrix} \omega_x(t) \\ \omega_y(t) \\ \omega_z(t) \end{bmatrix} = \begin{bmatrix} \omega_T \cos[\omega_{nb}(t-t_0)+\phi_0] \\ \omega_T \sin[\omega_{nb}(t-t_0)+\phi_0] \\ \omega_{\phi_0} \end{bmatrix} $$

3 American Institute of Aeronautics and Astronautics
where $\omega_T$ is the transverse component of angular velocity, which is constant in magnitude but circles around the $z$ axis, and $\omega_{zo}$ is the constant component along the $z$ axis. The rate at which $\omega_T$ circles the $z$ axis is the body-axes nutation rate, $\omega_{nb}$, and the phase shift $\phi_0$ is included to account for the location of the spin vector in the x-y plane at time $t_0$. In the inertial frame, the angular velocity nutates around the angular momentum vector $\mathbf{h}_m$ with the inertial nutation rate $\omega_{in}$. The angular momentum vector is given in spacecraft and inertial coordinates by

$$h_{sc0} = I_{sc} \omega_0 \quad \text{and} \quad h_{in} = A^T(q_0) I_{sc} \omega_0$$

(7)

Note that the diagonal structure of $I_{sc}$ makes Eq. (7) easily invertible. The two nutation rates are given by the expressions

$$\omega_{nb} = [1 - (I_S / I_T)] \omega_{z0}$$

(8)

$$\omega_{in} = \frac{\|h_{in}\|}{I_T} = \frac{\|h_{sc0}\|}{I_T} = \frac{\sqrt{I_T^2 \omega_T^2 + I_S^2 \omega_{zo}^2}}{I_T}$$

(9)

The quaternion state transition matrix for this restricted case can be obtained in closed form as a function of $\omega_0$, and it contains only simple trigonometric functions. Equivalently, one can find the quaternion for the rotation that transitions the spacecraft from its attitude at one time to its new attitude at another time. The quaternion state transition matrix is just the matrix that arises when quaternion multiplication is used to multiply the transition quaternion by the original quaternion. To find the transitional rotation in the restricted case, it is useful to view the problem dynamics as a body cone rolling on a space cone. The angular velocity rotates around the body $z$ axis in body coordinates, and the body $z$ axis rotates around the angular momentum vector as viewed in inertial space in a way that keeps the body $z$ axis, the angular momentum vector, and the angular velocity vector all in the same plane. The solution can be written in terms of the angular velocity at any time $t$. In the following development it is written in terms of the initial angular velocity or angular momentum. The solution is given in Refs. 5 and 6. It takes the form:

$$\Phi(t, t_0; \omega_0) = \begin{bmatrix}
q_{\phi_4}(t-t_0, \omega_0) & q_{\phi_3}(t-t_0, \omega_0) & -q_{\phi_2}(t-t_0, \omega_0) & q_{\phi_1}(t-t_0, \omega_0) \\
-q_{\phi_3}(t-t_0, \omega_0) & q_{\phi_4}(t-t_0, \omega_0) & q_{\phi_1}(t-t_0, \omega_0) & q_{\phi_2}(t-t_0, \omega_0) \\
q_{\phi_2}(t-t_0, \omega_0) & -q_{\phi_4}(t-t_0, \omega_0) & q_{\phi_3}(t-t_0, \omega_0) & q_{\phi_1}(t-t_0, \omega_0) \\
-q_{\phi_1}(t-t_0, \omega_0) & -q_{\phi_3}(t-t_0, \omega_0) & -q_{\phi_2}(t-t_0, \omega_0) & q_{\phi_4}(t-t_0, \omega_0)
\end{bmatrix}$$

(10)

where

$$\begin{bmatrix}
q_{\phi_1}(t-t_0, \omega_0) \\
q_{\phi_2}(t-t_0, \omega_0) \\
q_{\phi_3}(t-t_0, \omega_0) \\
q_{\phi_4}(t-t_0, \omega_0)
\end{bmatrix} = q_\phi(t-t_0, \omega_0) = q_s(t-t_0, \omega_0) \otimes q_h(t-t_0, \omega_0)$$

(11)

The two quaternions $q_s$ and $q_h$ parameterize two rotations that take the spacecraft from its attitude at $t_0$ to its attitude at $t$. $q_s$ is a rotation about the body frame $z$ axis at the body-axes nutation rate $\omega_{nb}$, and $q_h$ is a rotation about $h$ at the inertial nutation rate $\omega_{in}$. The “$\otimes$” symbol represents standard quaternion multiplication, with the convention that $A( q_s \otimes q_h ) = A(q_s) A(q_h)$. These two quaternions are given by:

$$q_s(t-t_0, \omega_0) = \begin{bmatrix}
0 \\
0 \\
\sin\{0.5(t-t_0)\omega_{ns}\} \\
\cos\{0.5(t-t_0)\omega_{ns}\}
\end{bmatrix}$$

(12)
The same quaternion state transition matrix can be written in terms of angular momentum by substituting Eqs. (8) and (9) (or the component-wise equivalents of Eq. (9)) into Eqs. (12) and (13). The new expressions are:

\[
q_s(t-t_0, h_{w0}) = \begin{bmatrix}
0 \\
0 \\
\sin\left(\frac{(t-t_0)}{2}\left(\frac{I_T - I_S}{I_T I_S}\right)h_{w03}\right) \\
\cos\left(\frac{(t-t_0)}{2}\left(\frac{I_T - I_S}{I_T I_S}\right)h_{w03}\right)
\end{bmatrix}
\]

(14)

\[
q_b(t-t_0, h_{w0}) = \begin{bmatrix}
h_{w0} \\
\|h_{w0}\| \\
\frac{(t-t_0)}{2}\frac{\hat{h}_{w0}}{\|h_{w0}\|} \\
\cos\left(\frac{(t-t_0)}{2}\|h_{w0}\|\right)
\end{bmatrix}
\]

(15)

The resulting transition quaternion is of the form given in Ref. 5:

\[
\begin{bmatrix}
q_{b0}(t-t_0, h_{w0}) \\
q_{b2}(t-t_0, h_{w0}) \\
q_{b3}(t-t_0, h_{w0}) \\
q_{b4}(t-t_0, h_{w0})
\end{bmatrix} = \begin{bmatrix}
\hat{h}_{w01}\cos\alpha\sin\beta + \hat{h}_{w02}\sin\alpha\sin\beta \\
\hat{h}_{w02}\cos\alpha\sin\beta - \hat{h}_{w01}\sin\alpha\sin\beta \\
\hat{h}_{w03}\cos\alpha\sin\beta + \sin\alpha\cos\beta \\
\cos\alpha\cos\beta - \hat{h}_{w03}\sin\alpha\sin\beta
\end{bmatrix}
\]

(16)

where

\[
\alpha = \frac{(t-t_0)(I_T - I_S)}{(2I_T I_S)}h_{w03}
\]

(17)

\[
\beta = \frac{(t-t_0)\|h_{w0}\|}{(2I_T)}
\]

(18)

\[
\hat{h}_{w0} = \frac{h_{w0}}{\|h_{w0}\|}
\]

(19)

These substitutions can be made in Eq. (16) in order to show the exact dependence of the rotation quaternion on \( h_{w0} \), but the resulting complicated formula has been omitted because it provides no useful insights.

The resulting quaternion state transition matrix contains only ordinary trigonometric functions. It would clearly be very difficult to solve the expressions it contains for any of the unknowns, whether in angular momentum or angular velocity form. Other parameterizations of this matrix exist. It is not clear that any of them provide a simpler means to solve the axially-symmetric restricted problem.

### IV. Partial Analytical Solutions

The three-observation case, because it is not overdetermined, reduces to an exercise in nonlinear equation solving. Given three vector measurements and assuming all measurement errors are zero, Ref. 5 demonstrates that the general case is locally observable. The steps used to solve the equations, however, depend on the choice of...
unknown parameters. In this section, four approaches are outlined for choosing a set of parameters and developing an appropriate set of nonlinear equations.

A. Aliasing

Any technique that relies on measurements obtained at different times must address the issue of aliasing. For a given angular momentum, the measurements must be taken with some minimum frequency in order to capture all information necessary for a solution. In the restricted case addressed in this paper, the three measurement times \( t_0 \), \( t_1 \), and \( t_2 \) are given and the angular momentum is unknown. Thus the issue of aliasing can be restated: For a given sampling rate, what is the maximum angular momentum (or equivalently, angular velocity) for which there are no aliasing concerns?

The Nyquist-Shannon sampling theorem states that the sampling frequency must be at least twice the value of any frequency component in the original signal that one wishes to reconstruct. In the attitude determination problem considered here, the spacecraft rotation is characterized by two frequency components: \( \omega_{nb} \) from Eq. (8) and \( \omega_{ns} \) from Eq. (9). This results in two constraints on the time between measurements given these angular rates, or alternatively, on two constraints on the angular velocity given some maximum measurement spacing. To avoid aliasing, the following must hold:

\[
\omega_{ns} < \frac{1}{4} \omega_{\text{amp}} \quad (20a)
\]
\[
\omega_{nb} < \frac{1}{4} \omega_{\text{amp}} \quad (20b)
\]

where \( \omega_{\text{amp}} = 2\pi/\Delta t \) is the minimum sampling rate for any two consecutive measurements. Substituting Eqs. (7) - (9) into Eqs. (20) and rearranging gives the following equivalent conditions on angular momentum:

\[
\| \dot{h}_i(t) \| = \left| I_T\omega_{\text{amp}} \right| < \frac{I_T\omega_{\text{amp}}}{2} = \frac{I_T\pi}{\Delta t} = \| \dot{h}_i(t) \|_{\text{max}} \quad (21a)
\]
\[
\left| h_3(t) \right| = \left| I_S\omega_{\text{amp}} \right| = \frac{I_S I_T}{\left| I_T - I_S \right|} \omega_{nb} < \frac{I_S I_T}{\left| I_T - I_S \right|} \frac{\omega_{\text{amp}}}{2} = \frac{I_S I_T}{\left| I_T - I_S \right|} \frac{\pi}{\Delta t} = \frac{I_S}{\left| I_T - I_S \right|} \| \dot{h}_i(t) \|_{\text{max}} = \left| h_3(t) \right|_{\text{max}} \quad (21b)
\]

If \( I_S > I_T \) (a major-axis spinner), then Eq. (21a) will always be more restrictive and Eq. (21b) can be ignored. On the other hand, if \( I_T > I_S \) (a minor-axis spinner), then the specific values of \( I_T \) and \( I_S \) and the nutation angle may all be necessary in order to determine which constraint will be violated first. While solutions may be possible even if these conditions are not satisfied, they provide a good check to ensure that aliasing has not occurred. Thus, this paper’s algorithms will restrict their attention to solutions that satisfy these limits.

B. A Useful Attitude Parameterization

Before covering these four approaches, one additional result is required. If there are no errors associated with the unit-length measurement vector \( b_i \), then the single measurement/reference pair \{ \( b_i, r_i \) \} partially constrains the quaternion solution at time \( t_i \). Each matrix \( K_i \) individually has two identical maximum eigenvalues, and two corresponding eigenvectors that are orthogonal to each other and that span the solution subspace. This same subspace is spanned by the two orthogonal, unit-normalized quaternions \(^8\)

\[
q_{is} = \left\{ \begin{array}{c}
\frac{1}{\sqrt{2(1 + b_i^T r_i)}} \begin{bmatrix} b_i \times r_i \\
(1 + b_i^T r_i) \end{bmatrix} \\
\sqrt{2(1 + b_i^T r_i)} c_i \\
0
\end{bmatrix} \quad \text{if } -1 < b_i^T r_i \\
0 \quad \text{if } -1 = b_i^T r_i
\end{array} \right. \quad (22a)
\]
\[ q_{ji} = \begin{bmatrix} b_j \\ 0 \end{bmatrix} \otimes q_{ai} \]  

(22b)

where \( c_i \) in Eq. (22a) is any unit direction vector orthogonal to \( b_j \). Any quaternion that represents a valid attitude solution given the single measurement pair can be written without loss of generality as the following linear combination of these two quaternions:

\[ q_{opt,j} = q_{ai} \cos \theta_i + q_{ji} \sin \theta_i \]  

(23)

Therefore, for each vector measurement, the set of possible attitude solutions is parameterized by the angle \( \theta_i \), which is half of the only remaining unknown rotation given \( r_i \) and \( b_j \). As parameterized here, it is a rotation about \( b_j \).

It is helpful to introduce two additional eigenvectors of the \( K_i \) matrix, which correspond to the two non-maximum eigenvalues. Call them \( q_{ci} \) and \( q_{di} \). They are orthogonal to each other and to \( q_{ai} \) and \( q_{bi} \). Therefore, these two quaternions are orthogonal to every possible solution quaternion.

C. Four Solution Approaches

Each of the following approaches begins with a set of unknowns that fully parameterize the solution and that will be determined in the approach. It then generates a set of independent equations in these unknowns. For each approach, vector measurements are assumed to be available at times \( t_0 \), \( t_1 \), and \( t_2 \). These four approaches do not exhaust the possibilities, but they provide a good sample of some reduced parameter sets that might be determined by various solution strategies.

1. Approach #1

This approach uses the angles \( \theta_0 \), \( \theta_1 \), and \( \theta_2 \), the angles used in attitude solutions in the form of Eq. (23), as three of the unknowns. Once determined, these angles can be used to produce the attitude quaternions at the three sample times:

\[ q(t_0) = q_{aj0} \cos \theta_0 + q_{aj0} \sin \theta_0 \]  

(24a)

\[ q(t_1) = q_{aj1} \cos \theta_1 + q_{aj1} \sin \theta_1 \]  

(24b)

\[ q(t_2) = q_{aj2} \cos \theta_2 + q_{aj2} \sin \theta_2 \]  

(24c)

The two transitional rotation quaternions can be computed as:

\[ q_{rot0} (\theta_0, \theta_1) = q(t_0) \otimes q^{-1}(t_1) \]  

(25a)

\[ q_{rot2} (\theta_2, \theta_1) = q(t_2) \otimes q^{-1}(t_1) \]  

(25b)

where the quaternion inverse \( q^{-1} \) is formed by negating the first three elements of \( q \). The rotation quaternions computed are also known as functions of \( h_{sc} \) from the quaternion state transition matrix solution in Eq. (16). Thus, one obtains the two equations

\[ q_\Phi (t_0-t_1, h_{sc}) = q_{rot0} (\theta_0, \theta_1) \]  

(26a)

\[ q_\Phi (t_2-t_1, h_{sc}) = q_{rot2} (\theta_2, \theta_1) \]  

(26b)

The result is two unit-normalized quaternion equations, or a total of eight scalar equations, in the six unknowns \( h_{sc1}, h_{sc2}, h_{sc3}, \theta_0, \theta_1, \) and \( \theta_2 \), where \( h_{sc1}, h_{sc2}, \) and \( h_{sc3} \) are the three components of the vector \( h_{sc} \). Although Eq. (26) seems like an overdetermined system of equations, it is not overdetermined because of the two normalization constraints. In theory, one could solve each set of scalar equations for \( h_{sc} \), Eq. (26a) for \( h_{sc} \) in terms of \( \theta_0 \) and \( \theta_1 \), and Eq. (26b) for \( h_{sc} \) in terms of \( \theta_2 \) and \( \theta_1 \). One would then set the two expressions for \( h_{sc} \) equal to each other.
These actions would result in three equations, one for each component of $h_{ac1}$, in the three unknowns $\theta_0$, $\theta_1$, and $\theta_2$. Note that, if the step of explicitly solving for $h_{ac1}$ is tractable, then this approach has the advantage of yielding three angular unknowns, which are limited to a range from $-\pi$ to $\pi$. Such a box could reasonably be searched by a fast computer.

It is difficult to get an explicit (as opposed to implicit) set of equations for the components of angular momentum. Suppose, however, that the time intervals between measurements are constant, that is, $(t_2-t_1) = (t_1-t_0)$. Then one can create a set of implicit equations that do not contain the variables $h_{ac11}$, $h_{ac12}$, and $h_{ac13}$. Consider the expressions for $q_0((t_0-t_1, h_{ac1})$ and $q_0((t_2-t_1, h_{ac1})$ given by Eq. (16):

\[
\begin{align*}
\begin{bmatrix}
q_{\phi_1}(t_2-t_1, h_{ac1}) \\
q_{\phi_2}(t_2-t_1, h_{ac1}) \\
q_{\phi_3}(t_2-t_1, h_{ac1}) \\
q_{\phi_4}(t_2-t_1, h_{ac1})
\end{bmatrix} &=
\begin{bmatrix}
\dot{h}_{ac11} \cos \alpha_0 \sin \beta_0 + \dot{h}_{ac12} \sin \alpha_0 \sin \beta_0 \\
\dot{h}_{ac12} \cos \alpha_0 \sin \beta_0 - \dot{h}_{ac11} \sin \alpha_0 \sin \beta_0 \\
\dot{h}_{ac13} \cos \alpha_0 \sin \beta_0 + \sin \alpha_0 \cos \beta_0 \\
\cos \alpha_0 \cos \beta_0 - \dot{h}_{ac13} \sin \alpha_0 \sin \beta_0
\end{bmatrix}
\end{align*}
\]

\begin{align*}
\begin{bmatrix}
q_{\phi_1}(t_0-t_1, h_{ac1}) \\
q_{\phi_2}(t_0-t_1, h_{ac1}) \\
q_{\phi_3}(t_0-t_1, h_{ac1}) \\
q_{\phi_4}(t_0-t_1, h_{ac1})
\end{bmatrix} &=
\begin{bmatrix}
\dot{h}_{ac11} \cos \alpha_0 \sin \beta_0 + \dot{h}_{ac12} \sin \alpha_0 \sin \beta_0 \\
\dot{h}_{ac12} \cos \alpha_0 \sin \beta_0 - \dot{h}_{ac11} \sin \alpha_0 \sin \beta_0 \\
\dot{h}_{ac13} \cos \alpha_0 \sin \beta_0 + \sin \alpha_0 \cos \beta_0 \\
\cos \alpha_0 \cos \beta_0 - \dot{h}_{ac13} \sin \alpha_0 \sin \beta_0
\end{bmatrix}
\end{align*}

(27a)

(27b)

where $\alpha_0 = (t_2-t_1)(I_z-I_0)/(2I_0I_z)$ and $\beta_0 = (t_2-t_1)|h_{ac1}|/(2I_0)$, as in Eqs. (17) and (18). In the case of equal time steps, $\alpha_2 = -\alpha_0$ and $\beta_2 = -\beta_0$. Thus Eq. (27b) can be re-written as:

\[
\begin{align*}
\begin{bmatrix}
q_{\phi_1}(t_2-t_1, h_{ac1}) \\
q_{\phi_2}(t_2-t_1, h_{ac1}) \\
q_{\phi_3}(t_2-t_1, h_{ac1}) \\
q_{\phi_4}(t_2-t_1, h_{ac1})
\end{bmatrix} &=
\begin{bmatrix}
-\dot{h}_{ac11} \cos \alpha_0 \sin \beta_0 + \dot{h}_{ac12} \sin \alpha_0 \sin \beta_0 \\
-\dot{h}_{ac12} \cos \alpha_0 \sin \beta_0 - \dot{h}_{ac11} \sin \alpha_0 \sin \beta_0 \\
-\dot{h}_{ac13} \cos \alpha_0 \sin \beta_0 - \sin \alpha_0 \cos \beta_0 \\
\cos \alpha_0 \cos \beta_0 - \dot{h}_{ac13} \sin \alpha_0 \sin \beta_0
\end{bmatrix}
\end{align*}
\]

(28)

By comparing Eqs. (27a) and (28), several relationships become evident. First, the sum of the squares of the first two elements of each quaternion is the same for both rotations:

\[
[q_{\phi_1}(t_0-t_1, h_{ac1})]^2 + [q_{\phi_2}(t_0-t_1, h_{ac1})]^2 - [q_{\phi_1}(t_2-t_1, h_{ac1})]^2 - [q_{\phi_2}(t_2-t_1, h_{ac1})]^2 = 0
\]

(29)

This relationship can be shown by straightforward algebra and the application of trigonometric identities, and it means that the lengths of the vectors composed of the first two elements of the two quaternions are equal. No algebra is necessary to show two more equations:

\[
q_{\phi_1}(t_0-t_1, h_{ac1}) + q_{\phi_3}(t_2-t_1, h_{ac1}) = 0
\]

(30)

\[
q_{\phi_2}(t_0-t_1, h_{ac1}) - q_{\phi_4}(t_2-t_1, h_{ac1}) = 0
\]

(31)

All three of Eqs. (29) - (31) hold true for any values of $h_{ac11}$, $h_{ac12}$, and $h_{ac13}$. Substitution of $q_{rot1}(\theta_0, \theta_1)$ and $q_{rot2}(\theta_2, \theta_1)$ into Eqs. (29) - (31) via Eqs. (26) yields a set of three equations in the unknowns $\theta_0$, $\theta_1$, and $\theta_2$:

\[
[q_{rot1,0}(\theta_0, \theta_1)]^2 + [q_{rot0,2}(\theta_0, \theta_1)]^2 - [q_{rot2,1}(\theta_2, \theta_1)]^2 - [q_{rot2,1}(\theta_2, \theta_1)]^2 = 0
\]

(32)

\[
q_{rot1,3}(\theta_0, \theta_1) + q_{rot2,3}(\theta_2, \theta_1) = 0
\]

(33)

\[
q_{rot1,4}(\theta_0, \theta_1) - q_{rot2,1}(\theta_2, \theta_1) = 0
\]

(34)
Unfortunately, Eqs. (32) - (34) are not completely independent. This result comes from the normalization constraint on the quaternions. Equation (32) requires that the first two element vectors of the quaternions have the same length for each rotation. In Eq. (33), the third elements are forced to have the same magnitude, and Eq. (34) imposes the same condition on the fourth elements. Taken together, these three equations require the two quaternions to have equal lengths, thus merely reiterating the state already determined by the unit normalization constraints. Therefore, additional equations would be required to obtain locally unique solutions.

Approach #1, therefore, needs additional development in order to yield a fully determined set of three nonlinear equations in three unknowns. It has been discussed here for two reasons. First, it constitutes a generalization of the method that has been used in Ref. 5 to develop its closed-form solution for its first extended Wahba problem. Second, some of this approach’s methods contribute to later approaches.

2. Approach #2

This approach begins in the same way as Approach #1, by assuming that the three $\theta$ angles are known and by using them to compute the two rotation quaternions $q_{\theta_01}(\theta_0, \theta_1)$ and $q_{\theta_021}(\theta_2, \theta_1)$. These two expressions are then equated, as in Approach #1, with the rotation quaternions given by Eq. (16) in terms of body-frame angular momentum at time $t_1$. Applying Eq. (16) to the time intervals $t_0-t_1$ and $t_2-t_1$, the first two elements of the respective rotation quaternions can be written:

\[
\begin{bmatrix}
q_{\theta_01}(t_0-t_1, h_{w1}) \\
q_{\theta_02}(t_0-t_1, h_{w1})
\end{bmatrix} = \sin \beta_{\theta_01} \begin{bmatrix}
\cos \alpha_{\theta_01} & \sin \alpha_{\theta_01} & \hat{h}_{w11} \\
-\sin \alpha_{\theta_01} & \cos \alpha_{\theta_01} & \hat{h}_{w12}
\end{bmatrix}
\]

\[
\begin{bmatrix}
q_{\theta_21}(t_2-t_1, h_{w1}) \\
q_{\theta_22}(t_2-t_1, h_{w1})
\end{bmatrix} = \sin \beta_{\theta_21} \begin{bmatrix}
\cos \alpha_{\theta_21} & \sin \alpha_{\theta_21} & \hat{h}_{w11} \\
-\sin \alpha_{\theta_21} & \cos \alpha_{\theta_21} & \hat{h}_{w12}
\end{bmatrix}
\]

(35a)

(35b)

In this form, one can see that the first two elements of each rotation quaternion are just the first two elements of the unit angular momentum vector, scaled by $\sin \beta$ and rotated by an angle $\alpha$. The rotation in both cases occurs around the body-frame z axis and within the body-frame x-y plane. It is interesting to note that the direction of the vector in Eq. (35a) lies along the projection of the angular momentum vector onto the x-y plane at time $(t_0+t_1)/2$. Similarly, the vector in Eq. (35b) is parallel to the x-y projection of the angular momentum vector at time $(t_1+t_2)/2$. This fact provides a means to determine the body axes nutation rate $\omega_{nb}$, because these projections rotate at that rate. Thus,

\[
\omega_{nb}(\theta_0, \theta_1, \theta_2) = \pm \frac{2 \arccos(\hat{q}_{\theta_01} \cdot \hat{q}_{\theta_21})}{(t_2-t_0)}
\]

(36)

where $\hat{q}_{\theta_01}$ is the unit vector parallel to the first two elements of $q_{\theta}(t_0-t_1, h_{w1})$, and $\hat{q}_{\theta_21}$ is the unit vector parallel to the first two elements of $q_{\theta}(t_2-t_1, h_{w1})$. In practice, the sign uncertainty can be resolved with an iterative procedure, but for the sake of simplicity in the derivation it will be assumed positive. Note that there is an integer ambiguity associated with the arccosine function in Eq. (36). If the non-aliasing condition in Eq. (20b) is satisfied, $n_1$ is guaranteed to be zero, and thus it will be ignored in the following development. A short proof of this result is contained in Appendix A.

Equation (36) can be combined with Eqs. (17) and (35) to show that the angles $\alpha_{\theta_01}$ and $\alpha_{\theta_21}$ are

\[
\alpha_{\theta_01}(\theta_0, \theta_1, \theta_2) = \frac{\omega_{nb}}{2}(t_0-t_1) = \arccos(\hat{q}_{\theta_01} \cdot \hat{q}_{\theta_21}) \frac{(t_0-t_1)}{(t_2-t_0)}
\]

(37a)

\[
\alpha_{\theta_21}(\theta_0, \theta_1, \theta_2) = \frac{\omega_{nb}}{2}(t_2-t_1) = \arccos(\hat{q}_{\theta_01} \cdot \hat{q}_{\theta_21}) \frac{(t_2-t_1)}{(t_2-t_0)}
\]

(37b)
Next, these angles enter into Eq. (12), the quaternion rotation about the body z axis:

\[
q_{\alpha_0}(t_0-t; \theta_0, \theta_1, \theta_2) = \begin{bmatrix} 0 \\ 0 \\ \sin \alpha_0 \\ \cos \alpha_0 \end{bmatrix}
\] (38a)

\[
q_{\alpha_2}(t_2-t; \theta_0, \theta_1, \theta_2) = \begin{bmatrix} 0 \\ 0 \\ \sin \alpha_2 \\ \cos \alpha_2 \end{bmatrix}
\] (38b)

As discussed in Section III, the change in spacecraft attitude between any two times is known to be composed of two rotations, one about the body spin axis, here given by Eqs. (38), and one about the angular momentum vector. This second rotation about the angular momentum vector can thus be obtained for each time interval from Eq. (11), by using quaternion inversion and quaternion multiplication.

\[
q_{\alpha_0}(t_0-t, \theta_0, \theta_1, \theta_2) = q_{\alpha_0}^{-1}(t_0-t, \theta_0, \theta_1, \theta_2) \otimes q_{\alpha_0}(t_0-t, \theta_0, \theta_1, \theta_2)
\] (39a)

\[
q_{\alpha_2}(t_2-t, \theta_0, \theta_1, \theta_2) = q_{\alpha_2}^{-1}(t_2-t, \theta_0, \theta_1, \theta_2) \otimes q_{\alpha_2}(t_2-t, \theta_0, \theta_1, \theta_2)
\] (39b)

Equations (39) can be substituted into the expression for \(q_h\) given in Eq. (15).

\[
q_{\alpha_0}(t_0-t, \theta_0, \theta_1, \theta_2) = q_{\alpha_0}(t_0-t, h_{\alpha_1}) = \begin{bmatrix} h_{\alpha_1} \\ \|h_{\alpha_1}\| \end{bmatrix} \sin \left( \frac{(t_0 - t_1) \|h_{\alpha_1}\|}{2I_T} \right) \\

\cos \left( \frac{(t_0 - t_1) \|h_{\alpha_1}\|}{2I_T} \right)
\] (40a)

\[
q_{\alpha_2}(t_2-t, \theta_0, \theta_1, \theta_2) = q_{\alpha_2}(t_2-t, h_{\alpha_1}) = \begin{bmatrix} h_{\alpha_1} \\ \|h_{\alpha_1}\| \end{bmatrix} \sin \left( \frac{(t_2 - t_1) \|h_{\alpha_1}\|}{2I_T} \right) \\

\cos \left( \frac{(t_2 - t_1) \|h_{\alpha_1}\|}{2I_T} \right)
\] (40b)

Finally, the theoretical expressions given by Eqs. (40) allow the formation of several equations in terms of the three \(\theta\) angles. If the two time steps have equal length, then \(q_{\alpha_0}\) and \(q_{\alpha_2}\) are quaternion inverses of each other; that is, the first three elements have opposite signs, and the fourth elements are equal. In this case, the expression

\[
q_{\alpha_0} \otimes q_{\alpha_2} = q_{\alpha_0} \otimes q_{\alpha_0}^{-1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}
\] (41)

gives a set of four equations in \(\theta_0\), \(\theta_1\), and \(\theta_2\), of which not more than three are independent because of the unit length constraint.
If the time steps are not equal, other properties must be exploited. The first three elements of each quaternion constitute a unit vector in the direction of the angular momentum vector, scaled by the sine expression. Because the vectors for both quaternions are parallel to each other even if the scaling is different,

\[
\begin{pmatrix}
3 \\
1213 \\
101
\end{pmatrix} = \times \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\]  

(42)

Another equation comes from relating the arguments of the sines and cosines within \( q_{h01} \) and \( q_{h21} \). If these arguments are called \( \beta_{01} \) and \( \beta_{21} \), then

\[
\frac{\beta_{01}}{t_0 - t_1} = \frac{\beta_{21}}{t_2 - t_1}
\]  

(43)

The \( \beta \) arguments can be extracted with a two-argument arctangent. Another ambiguity appears in this calculation, but as before it can be shown that if the non-aliasing conditions are satisfied, the ambiguity will be identically zero. Appendix B contains the derivation of this condition. \( \beta_{01} \) and \( \beta_{21} \) are indirectly related to \( \alpha_{01} \) and \( \alpha_{21} \) from Eqs. (37), because both sets of parameters contain information about the magnitude of the angular momentum components. Any valid solution must maintain consistency between the corresponding \( \alpha \) and \( \beta \) values.

Two sets of four equations have thus been specified: if the time steps are constant, one can use Eqs. (41), and if the time steps are not equal, one can use Eqs. (42) and Eq. (43). Note that each set of four equations has at most three independent components because of the unit length constraints on the quaternions and on the direction vector \( \hat{h}_{h1} \). The three angular unknowns define a limited solution space, but the magnitude of the spacecraft angular velocity cannot be determined fully without specifying two integer ambiguities.

3. Approach #3

The idea behind the third approach is to choose unknown parameters that capture the initial attitude and rate succinctly, and then apply the known dynamics equations to propagate attitude and rate forward in time. One forms equations by constraining the solution parameters and the dynamically propagated quantities to be consistent with the vector attitude measurements.

One can start with \( \theta_0 \) and \( \theta_1 \) as two of the unknowns that are used to parameterize the solution, and one can use Eqs. (24a), (24b) and (25a) to express \( q_{rot01} \) in terms of \( \theta_0 \), and \( \theta_1 \). Recall from Eq. (11) that each rotation between sample times is composed of a rotation about the body-frame \( z \) axis and a rotation about the angular momentum axis in the inertial frame. The third chosen unknown is half the rotation about the body \( z \) axis from time \( t_1 \) to time \( t_0 \), as in Eq. (12):

\[
\alpha = 0.5\omega_{nb}(t_0-t_1)
\]  

(44)

Note that this \( \alpha \) is identical to the \( \alpha \) defined in Eq. (17) except for the choice of time span. One expresses the rotation about the body \( z \)-axis, \( q_r \), in terms of this angle:

\[
q_{rot}(\alpha) = \begin{bmatrix}
0 \\
0 \\
\sin(0.5(t_0-t_1)\omega_{nb}) \\
\cos(0.5(t_0-t_1)\omega_{nb})
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
\sin(\alpha) \\
\cos(\alpha)
\end{bmatrix}
\]  

(45)
The unknown parameters $\theta_0$ and $\theta_1$ contain information about the attitude at the first two measurement times, and the parameter $\alpha$ partially describes the angular rate. The process of deriving three equations in the three unknowns starts by extracting $h_{x_1}$, the angular momentum at time $t_1$, from the way in which the partially specified rotation takes the attitude from $q_0$ to $q_1$. First, one multiplies $q_{w0}$ by the inverse of $q_{01}(\alpha)$ in order to derive an expression for $q_{01}(t_0-t_1)$, which is the rotation about the angular momentum vector, in terms of $\theta_0$, $\theta_1$, and $\alpha$:

$$q_{01}(\theta_0, \theta_1, \alpha) = q_{w0}^{-1}(\alpha) \otimes q_{w01}^{-1} = q_{w0}^{-1}(\alpha) \otimes q(t_0; \theta_0) \otimes q^{-1}(t_1; \theta_1)$$ (46)

The quaternion $q_{01}(\theta_0, \theta_1, \alpha)$ is known to take the form of Eq. (15):

$$q_{01}(t_0-t_1, h_{x_1}) = \begin{bmatrix} h_{x_1} \|h_{x_1}\| \sin \left( \frac{(t_0-t_1) \|h_{x_1}\|}{2I_T} \right) \\ \cos \left( \frac{(t_0-t_1) \|h_{x_1}\|}{2I_T} \right) \end{bmatrix}$$ (47)

The equivalence of $q_{01}(\theta_0, \theta_1, \alpha)$ in Eq. (46) with $q_{01}(t_0-t_1, h_{x_1})$ in Eq. (47) makes it possible to calculate the unit direction vector $h_{x_1}/\|h_{x_1}\|$ in terms of $\theta_0$, $\theta_1$, and $\alpha$ simply by computing the unit direction vector that points in the same direction as the first three elements of $q_{01}(\theta_0, \theta_1, \alpha)$. One can also obtain the magnitude of the angular momentum. It can be determined by using a two-argument arctangent function to extract the argument of the sine and cosine in $q_{01}$, because the value of $\cos \{0.5(t_0-t_1)\|h_{x_1}\|/I_T\}$ is known from the fourth element of $q_{01}(\theta_0, \theta_1, \alpha)$ and the value of $\sin \{0.5(t_0-t_1)\|h_{x_1}\|/I_T\}$ is known from the magnitude of the vector that consists of the first three elements of $q_{01}(\theta_0, \theta_1, \alpha)$.

$$\arctan2 \left( \frac{\|q_{01}(t_0-t_1, h_{x_1})\|_3}{\|q_{01}(t_0-t_1, h_{x_1})\|_4} \right) = (t_0-t_1)\frac{\|h_{x_1}\|}{2I_T} + 2\pi n$$ (48)

for some integer $n$. Consequently,

$$\|h_{x_1}\| = \|h_{x_1}(t)\| = 2I_T \left( \arctan2 \left( \frac{\|q_{01}(t_0-t_1, h_{x_1})\|_3}{\|q_{01}(t_0-t_1, h_{x_1})\|_4} \right) - 2\pi n \right) / (t_0-t_1)$$ (49)

Once again, the ambiguity $n$ can be shown to equal zero provided the non-aliasing conditions have been met. The proof of this result is algebraically identical to the proof in Appendix B for the second Approach #2 ambiguity. For the remainder of this discussion, the integer ambiguity $n$ will be assumed to be zero on this basis and thus ignored.

At this point, the first of the three nonlinear equations can be formed by relating the parameter $\alpha$ to the expression derived for $h_{x_1}$ in terms of $\theta_0$, $\theta_1$, $\alpha$. Equation (7) allows the third component of $h_{x_1}$ to be written in terms of the third component of angular velocity, $\omega_{z0}$. Then $\alpha$ relates to $\omega_{z0}$ as follows:

$$\alpha = \omega_{z0}(\theta_0, \theta_1, \alpha) \left( \frac{I_T-I_S}{I_S} \right) (t_0-t_1) = \frac{h_{x_13}(\theta_0, \theta_1, \alpha)}{I_S} \left( \frac{I_T-I_S}{I_T} \right) (t_0-t_1)$$ (50)

This equation constitutes one of the three equations that must be solved for the three unknowns $\theta_0$, $\theta_1$, and $\alpha$. 

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Next, one can use the derived expression for $h_{s1}$ along with $\alpha$ to propagate the spacecraft attitude forward in time from $q_1$. The rotation about the body spin axis during this time interval occurs at the same rate as before, and is therefore given by

$$q_{i2} (\alpha) = \begin{bmatrix}
0 \\
0 \\
\sin \alpha \frac{(t_2-t_i)}{(t_0-t_i)} \\
\cos \alpha \frac{(t_2-t_i)}{(t_0-t_i)}
\end{bmatrix}$$  \hspace{1cm} (51)

Similarly, one forms $q_{i2}$ by substituting $h_{s1}(\theta_0, \theta_1, \alpha)$, as derived from Eqs. (47) - (49), into Eq. (15). The composite rotation then propagates the attitude to time $t_2$:

$$q_2 = q(t_2) = q_{i2} (\alpha) \otimes q_{i1} [t_2-t_i; h_{s1}(\theta_0, \theta_1, \alpha)] \otimes q(t_i; \theta_i)$$  \hspace{1cm} (52)

This equation gives the quaternion at time $t_2$ in terms of $\theta_0$, $\theta_1$, and $\alpha$.

In order to construct two more equations in the three unknowns, the attitude $q_2$ is constrained to be consistent with the vector attitude measurement at $t_2$. This constraint is accomplished by requiring the attitude to have the form given in Eqs. (22) - (23). Consequently, the quaternion $q_2$ is orthogonal to both $q_{c2}$ and $q_{d2}$, the eigenvectors of $K_2$ corresponding to its non-maximum eigenvalues. This fact yields two additional equations:

$$q_{c2}^T q(t_2) = 0$$  \hspace{1cm} (53a)
$$q_{d2}^T q(t_2) = 0$$  \hspace{1cm} (53b)

There are now three equations, Eqs. (50), (53a), and (53b), and three unknowns. The three-dimensional angular parameter space is limited to the range from $-\pi$ to $\pi$ for $\theta_0$ and $\theta_1$ and to the range from $-l\pi$ to $l\pi$ for $\alpha$, where $l$ is the maximum number of cycles of aliasing of the body-axis nutation rate from sample time $t_0$ to sample time $t_1$.

4. Approach #4

The fourth approach differs from the others in that it defines a system of equations with the three body axes components of angular momentum at time $t_1$ as unknowns. One additional attitude parameter also enters into the equations, but it can be eliminated easily. Unlike the angular attitude parameter in the previous approaches, angular momentum parameters are not by their nature limited to a finite range of values. This approach thus avoids some ambiguities, but at the expense of introducing a theoretically infinite range within which solutions could lie.

Assume that angular momentum is available at time $t_1$ and is given by

$$h_{sc1} = h_{sc}(t_1) = [h_{sc11} \ h_{sc12} \ h_{sc13}]^T$$  \hspace{1cm} (54)

Furthermore, the optimal quaternion at $t_1$ can be written according to Eq. (23) as

$$q(t_1) = q_{o1} \cos \theta_1 + q_{p1} \sin \theta_1$$  \hspace{1cm} (55)

Compute the quaternion rotation $q_{o}(t_0-t_i, h_{sc1})$ from time $t_1$ to time $t_0$ by employing Eq. (16), and use it to find $q(t_0)$ as a function of $h_{sc1}$ and $\theta_1$:

$$q(t_0) = q_{o}(t_0-t_i, h_{sc1}) \otimes [q_{o1} \cos \theta_1 + q_{p1} \sin \theta_1]$$  \hspace{1cm} (56)
Next, compute $q_{α0}$ and $q_{β0}$ by using Eq. (22), and then use any appropriate method to compute $q_{c0}$ and $q_{d0}$ such that the matrix $[q_{α0}, q_{β0}, q_{c0}, q_{d0}]$ is orthonormal. Suitable methods include singular value decomposition and orthogonal/upper triangular factorization (QR factorization). Both $q_{c0}$ and $q_{d0}$ are perpendicular to any quaternion solution, so they can be used to form the two equations

$$0 = q_{α0}^T q(t_0)$$  \hspace{1cm} (57a)  \\
$$0 = q_{β0}^T q(t_0)$$  \hspace{1cm} (57b)

where $q(t_0)$ is a function of $h_{sc11}$, $h_{sc12}$, $h_{sc13}$, and $θ_1$. Satisfaction of these two equations will guarantee that $q(t_0)$ as determined from $h_{sc1}$ and $θ_1$ using Eq. (56) can be re-cast into the form in Eq. (23), with $i = 0$, for some value of $θ_0$. One could also write these two equations in matrix form as

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} μ_1(h_{sc1}) \\ μ_2(h_{sc1}) \end{bmatrix} \cos θ_1 + \begin{bmatrix} η_1(h_{sc1}) \\ η_2(h_{sc1}) \end{bmatrix} \sin θ_1$$  \hspace{1cm} (58)

where

$$μ_1(h_{sc1}) = q_{α0}^T [q_{α}(t_0-τ_1, h_{sc1}) \otimes q_{a1}]$$  \hspace{1cm} (59a)  \\
$$μ_2(h_{sc1}) = q_{β0}^T [q_{β}(t_0-τ_1, h_{sc1}) \otimes q_{a1}]$$  \hspace{1cm} (59b)  \\
$$η_1(h_{sc1}) = q_{α0}^T [q_{α}(t_0-τ_1, h_{sc1}) \otimes q_{β1}]$$  \hspace{1cm} (59c)  \\
$$η_2(h_{sc1}) = q_{β0}^T [q_{β}(t_0-τ_1, h_{sc1}) \otimes q_{β1}]$$  \hspace{1cm} (59d)

and all are scalar nonlinear functions in the variables $h_{sc11}$, $h_{sc12}$, and $h_{sc13}$. Next, one repeats the whole process, except this time using

$$q(t_2) = q_{α}(t_2-τ_1, h_{sc1}) \otimes [q_{α1} \cos θ_1 + q_{β1} \sin θ_1]$$  \hspace{1cm} (60)

and one finds $q_{c2}$ and $q_{d2}$ using some suitable method to get the equations

$$0 = q_{c2}^T q(t_2)$$  \hspace{1cm} (61a)  \\
$$0 = q_{d2}^T q(t_2)$$  \hspace{1cm} (61b)

These equations can be written in matrix form of Eq. (58), and even stacked with that system of equations if desired. They take the form

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} μ_3(h_{sc1}) \\ μ_4(h_{sc1}) \end{bmatrix} \cos θ_1 + \begin{bmatrix} η_3(h_{sc1}) \\ η_4(h_{sc1}) \end{bmatrix} \sin θ_1$$  \hspace{1cm} (62)

where

$$μ_3(h_{sc1}) = q_{α2}^T [q_{α}(t_0-τ_1, h_{sc1}) \otimes q_{a1}]$$  \hspace{1cm} (63a)  \\
$$μ_4(h_{sc1}) = q_{β2}^T [q_{β}(t_0-τ_1, h_{sc1}) \otimes q_{a1}]$$  \hspace{1cm} (63b)
\[ \eta_i(h_{sc1}) = g_{sc}^T [q_0 \theta_i h_{sc1} \otimes q_{ji} ] \] (63c)

\[ \eta_i(h_{sc1}) = g_{sc}^T [q_0 (t_0 - t_i) h_{sc1} \otimes q_{ji} ] \] (63d)

The final result is a set of four equations in the four unknowns \( h_{sc1}, h_{sc12}, h_{sc13}, \) and \( \theta_i \), contained in the two pairs of stacked Eqs. (58) and (62). It is relatively simple to solve for \( \theta_i \) given the angular momentum by using any non-zero equation and the arctangent function, but the subsequent explicit solution for the optimal components of \( h_{sc1} \) would be extremely difficult. This complexity is because the angular momentum components enter the formulas for the rotation quaternions in a nonlinear fashion, as in Eq. (16), and furthermore they are not limited to a range from \(-\pi \) to \( \pi \) as is \( \theta_i \).

There is an alternative way to interpret Eqs. (58) and (62). Stacking the two sets of equations and rearranging gives

\[
\begin{bmatrix}
\mu_1(h_{sc1}) & \eta_1(h_{sc1}) \\
\mu_2(h_{sc1}) & \eta_2(h_{sc1}) \\
\mu_3(h_{sc1}) & \eta_3(h_{sc1}) \\
\mu_4(h_{sc1}) & \eta_4(h_{sc1})
\end{bmatrix}
\begin{bmatrix}
\cos \theta_i \\
\sin \theta_i
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\] (64)

From this matrix equation, it becomes clear that the values of \( h_{sc11}, h_{sc12}, h_{sc13}, \) and \( \theta_i \) that solve the equations must be such that the 4x2 matrix of \( \mu \)'s and \( \eta \)'s has the non-trivial nullspace \([\cos \theta_i \sin \theta_i]^T\). Thus the two columns of the 4x2 matrix differ only by scalar multiplication. Although this interpretation is interesting, it does not provide any obvious equation-solving benefits.

V. A General Numerical Approach and Other Considerations

A. Numerical Approach

For the general case of the second Wahba’s problem proposed in Ref. 5, a Taylor series approximation of the entire performance metric \( q_0^T K(\omega_0) q_0 \) can be used to attempt an optimization based on Newton’s method. This general procedure is outlined in Ref. 9. In the restricted (axially-symmetric, three measurement) case, however, a numerical approach can be devised that exploits knowledge of the system dynamics.

One of the most obvious numerical approaches is to start with one of the systems of nonlinear equations derived in Section IV and to use Newton’s method for nonlinear equation solving (as outlined in Section 4.7.6 of Ref. 9) to find a solution. For instance, if Eqs. (58) and (62) from Approach #4 in Section IV were used, then the problem would be one of trying to find \( h_{sc11}, h_{sc12}, h_{sc13}, \) and \( \theta_i \) to solve

\[
f(h_{sc11}, h_{sc12}, h_{sc13}, \theta_i) = \begin{bmatrix}
\mu_1(h_{sc1}) \\
\mu_2(h_{sc1}) \\
\mu_3(h_{sc1}) \\
\mu_4(h_{sc1})
\end{bmatrix} \cos \theta_i + \begin{bmatrix}
\eta_1(h_{sc1}) \\
\eta_2(h_{sc1}) \\
\eta_3(h_{sc1}) \\
\eta_4(h_{sc1})
\end{bmatrix} \sin \theta_i = \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\] (65)

Newton’s method consists of starting with a first guess of \( h_{sc11}, h_{sc12}, h_{sc13}, \) and \( \theta_i \), linearizing Eq. (65) about that guess, solving the linearized equation for increments \( \Delta h_{sc11}, \Delta h_{sc12}, \Delta h_{sc13}, \) and \( \Delta \theta_i \), using those increments to improve the guesses of \( h_{sc11}, h_{sc12}, h_{sc13}, \) and \( \theta_i \), and iterating.

Newton’s method can be forced to converge by considering the cost function

\[
J_f(h_{sc11}, h_{sc12}, h_{sc13}, \theta_i) = \frac{1}{2} f^T f
\] (66)
This cost function is non-negative, and any solution that causes this cost function to equal zero will exactly satisfy Eq. (65). It will also maximize the \( q \)-method performance metric of Eq. (4b), or minimize the equivalent cost function of the form in Eq. (1).

Newton’s method can be augmented with a step size selection along the search direction 
\[
[\Delta h_{s1} \ \Delta h_{s2} \ \Delta h_{s13} \ \Delta h_1]^T
\]
in order to guarantee convergence to a solution that locally minimizes the cost function in Eq. (66), but this approach may fail to globally minimize the cost function. Even if it does reach the global minimum of \( J_f = 0 \), there may be multiple attitude solutions that are globally optimal and that fit the data, as was the case for the solution obtained in Ref. 5 for the first extended Wahba’s problem.

B. Handling of Multiple Solutions

The problem of multiple globally optimal solutions cannot be solved except by obtaining more data, but it would be beneficial at least to know when multiple solutions may occur. In a general problem with the solution unconstrained, it might not be possible to address these concerns adequately, but for the approaches outlined in Section IV there may be a relatively simple answer. The key is that the solution spaces for the sets of equations derived in Section IV are not completely unconstrained. For example, the \( \theta \) angles are limited to a range from 0 to \( \pi \), with a physically equivalent \( \theta \) occurring between \( -\pi \) and 0. If the measurements are frequent enough relative to the spin rate that aliasing has not occurred, then some limits can also be placed on the magnitude of the angular velocity and angular momentum. The result in each case should be a finite three- or four-dimensional space wherein the solutions must lie. This space can be divided into a grid of points, and each point used as a starting point for the Gauss-Newton solution. If the grid spacing is sufficiently fine, then one or more starting points should converge to a solution that causes the cost function in Eq. (66) to be exactly zero, and thus globally optimizes the quaternion attitude solution. If there are multiple global minima, then the different global minima should be reached by applying Newton’s method starting from different of the grid points.

C. The Non-Uniqueness of the Quaternion Attitude Representation

The quaternion parameterization is global and non-singular, but it is not one-to-one; the same physical rotation is represented by \( q \) and \(-q\). Thus, it is possible to mean two different things when equating quaternions. \( q_A = q_B \) can mean that all four elements of \( q_A \) and \( q_B \) are equal, or it can mean that \( q_A \) and \( q_B \) represent the same rotation. Care must be taken to properly handle this distinction when deriving and implementing equations for the various solution approaches. In particular, this issue may arise for numerical solutions. Unfortunately, it is not always possible to just choose a quaternion sign convention and force all quaternions to conform. Suppose a quaternion \( q(\gamma) \) is constructed as a function of some parameter (or vector of parameters) \( \gamma \), and a method is constructed to invert the equations and determine \( \gamma(q) \). Even though \( q(\gamma) = -q(\gamma) \) in the equivalent rotation sense, it may often be that \( \gamma(q) \neq \gamma(-q) \). For instance, it might hold that \( \gamma(q) = \gamma(-q) + \pi \). A situation like this one can occur when a gradient-based search algorithm is implemented to try to solve one of the systems of equations derived in Section IV.

D. Independence of Reference Vectors

One of the results in Ref. 5 is that it is not necessary for all three reference vectors in the measurement series to be linearly independent in order for the restricted problem to be observable. Any two of the three reference vectors may be identical, as long as the third vector is different. At least one body axes measurement vector must also be different from the others to resolve the attitude rotation around that body axis. While theoretically two identical reference vectors is acceptable, it may be more problematic when a solution is implemented numerically. A change in the reference vectors used for measurements does not change the problem dynamics, but it may change the behavior of Eq. (66), the squared equation error cost function. Newton’s method relies on equation derivatives in order to converge to a solution. Consequently any change that affects the smoothness of Eq. (66) in the neighborhood of a solution may hinder a numerical solver.
Table 1. Parameters for truth model scenarios.

<table>
<thead>
<tr>
<th>Description</th>
<th>Scenario #1</th>
<th>Scenario #2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Symmetric about major axis</td>
<td>0.5227</td>
<td>4.0417</td>
</tr>
<tr>
<td>Symmetric about minor axis</td>
<td>.0967 rad/sec</td>
<td>.0764 rad/sec</td>
</tr>
<tr>
<td>$\Delta t$</td>
<td>15.0 sec</td>
<td>27.4 sec</td>
</tr>
<tr>
<td>Angle between $h_{sc}$ and $z$ axis</td>
<td>0.670°</td>
<td>88.9°</td>
</tr>
</tbody>
</table>

VI. Numerical Results for Solution Approaches

Numerical techniques like those described above, using Newton’s method and linearizing the equations at each step, were applied to the last three of the four approaches in Section IV. Approach #1 was not included because its solution was not fully determined by the derived equations. The procedure used was as follows: First, a truth model simulation was used to create a simulated sequence of vector measurements and the corresponding “truth” time histories of attitude and angular rate. Two dynamic scenarios were studied, including spacecraft symmetric about the major axis and the minor axis. The simulated measurements were sufficiently close in time to avoid aliasing. Table 1 lists some of the values used in the truth model simulations. Next, a valid solution subspace was defined by assuming knowledge only of the order of magnitude of angular momentum. Within this subspace, 1000 points were chosen at random to initialize the solution procedure, and Newton’s method was applied until the solution parameters converged to a global minimum, got trapped at a local minimum, or diverged to unreasonable parameter values. For those points which exactly solved the squared equation error cost function in Eq. (66) to the limits of numerical precision, additional calculations were performed to determine other solution quantities (such as angular rates or quaternions) not explicitly calculated as part of the solution to that approach. These solutions were compared to the “truth” values generated at the beginning of the procedure. In most cases, the number of global minima was greater than the number of physically identical solution points. It was found that the solution ambiguities could be resolved with one additional measurement at time $t_3$. To incorporate this measurement, the procedure described here was performed for the two overlapping measurement sets occurring at times $\{t_0, t_1, t_2\}$ and at times $\{t_1, t_2, t_3\}$. The correct solutions were found at the intersection of the solution sets for the two measurement sequences. The success of this technique indicates that the unrestricted problem (with more than three vector measurements) is globally observable, even if the restricted problem is only locally observable.

A. Numerical Results for Approach #2

When the procedure outlined above was performed for Approach #2, the numerical solution points indicated a family of solution curves within the $\{\theta_0, \theta_1, \theta_2\}$ space rather than a finite number of points. This result suggests that each set of four equations contains only two independent equations, rather than the expected three. Examination of the system of equations revealed that the numerical implementation had failed to maintain the necessary consistency between the $\alpha$ and $\beta$ parameters. In other words, the magnitudes of the rotations specified by $q_{h01}$ and $q_{h21}$ were not made to bear the proper functional relationship to the nutation rate $\omega_{nb}$. As a result, the equations were not sufficiently restrictive and only a one-dimensional subspace of solutions was determined. The two integer ambiguities inherent in Approach #2, as well as the non-uniqueness of the quaternion attitude parameterization, hindered the addition of the proper constraints to the numerical implementation. Figures 1a and 1b show the one-dimensional subspaces to which the solutions converged in blue, with the correct solutions marked in red. Note that there are eight physically identical correct solutions in the $\{\theta_0, \theta_1, \theta_2\}$ space. Scenario #2 in Fig. 1b seems to have converged to point solutions instead of curves, but actually the subspace has merely shrunk to form small rings of global minima intersecting the true solutions.
Figure 1a. Approach #2, Scenario #1 numerical solution subspace for $\{\theta_0, \theta_1, \theta_2\}$.

Figure 1b. Approach #2, Scenario #2 numerical solution subspace for $\{\theta_0, \theta_1, \theta_2\}$.
B. Numerical Results for Approach #3

Approach #3 met with greater success. It correctly determined the attitude and rate, with four mathematically different but physically identical solutions, but it also generated additional incorrect global minimum points. For sufficiently small simulated measurement errors, this solution ambiguity could be correctly resolved by incorporating an additional measurement as described above. Approximately 49% of the initial points for Scenario #1 converged to global minima, as opposed to 29% of the initial points for Scenario #2. However there were fewer incorrect minimum points for Scenario #2. The numerical solver was able to converge to the correct solutions even when any two of the three reference vectors were identical, although a smaller set of the initial points converged in these cases. Approach #3 did not, however, deal well with aliasing. If the ambiguity was not known in advance and explicitly included in the calculations, then incorrect, aliased solutions resulted and the numerical methods did not find the correct solutions at all. Figures 2a and 2b give the solution points in \{\theta_0, \theta_1, \alpha\} space for two overlapping measurement sets: \{t_0, t_1, t_2\} in blue and \{t_1, t_2, t_3\} in black. At the true values (shown in red), the blue and black points coincide.

![Global Minima for Approach #3, Scenario #1](image)

Figure 2a. Approach #3, Scenario #1 numerical solutions for \{\theta_0, \theta_1, \alpha\}.
C. Numerical Results for Approach #4

Approach #4 was the most promising in several respects. First, because the parameter space was related to angular momentum and not orientation, it avoided the multiplicity of physically identical solutions inherent in Approach #3. Second, although extra global minima were generated, it was sometimes possible to rule them out as valid solutions based solely on an unreasonably high or low angular momentum magnitude, without recourse to additional measurements. When multiple global minima occurred at similar angular momenta, the technique of re-applying the calculations to a different set of three measurements was able to identify the correct solution for cases with sufficiently small measurement errors. A final advantage was superior ability to handle aliasing. Because the parameters chosen already contained information about the angular rate, correct solutions were possible even if the measurements were spaced relatively far apart in time. To offset these advantages, Approach #4 required some a priori knowledge of limits on the angular momentum magnitude in order to avoid an infinite solution space in the numerical search procedure. Even when initializing a numerical solution with a small angular momentum, the Newton-Raphson search often diverged to unreasonably large values of $|h_{\text{sc}}|$. In fact, only 26% of the initial points for Scenario #1 and only 1% of the initial points for Scenario #2 converged to global minima. When $r(t_0) = r(t_1)$ or $r(t_0) = r(t_2)$, a smaller percentage of initial points converged. In the case where $r(t_1) = r(t_2)$, the cost function in the form of Eq. (66) behaved poorly enough to prevent convergence even with a very large number of initial points. Figures 3a and 3b show characteristic sets of global minima for two different but overlapping measurement sets for each of the two simulated scenarios.

Figure 2b. Approach #3, Scenario #2 numerical solutions for $\{\theta_0, \theta_1, \alpha\}$.  

![Figure 2b. Approach #3, Scenario #2 numerical solutions for $\{\theta_0, \theta_1, \alpha\}$](image-url)
Figure 3a. Approach #4, Scenario #1 numerical solutions for \( \{h_{sc11}, h_{sc12}, h_{sc13}\} \).

Figure 3b. Approach #4, Scenario #2 numerical solutions for \( \{h_{sc11}, h_{sc12}, h_{sc13}\} \).
VII. Summary and Conclusions

This paper has focused on the solution of a restricted form of an extended Wahba’s problems that seeks to estimate the initial attitude and angular rate of a spinning spacecraft. Because of the advantages of global optimality and convergence, this type of attitude estimation may be superior to standard EKF methods in many cases. The restricted problem considers the case of a rigid spacecraft with axial symmetry, a known moment of inertia, and zero external torque. It uses a single vector attitude measurement available at each of three distinct measurement times. Axial symmetry allows the quaternion state transition matrix to be written in a closed form involving only trigonometric functions. The problem is equivalent to solving six nonlinear equations in six unknowns.

Algebraic techniques have been used to reduce the restricted problem to a set of only three or four nonlinear equations in as many unknowns. This process is aided by a careful selection of unknown parameters. When aliasing does not occur, some approaches result in only three angular unknowns, so that the solution space is a three-dimensional rectangle with side lengths of $\pi$ or $2\pi$. A combination of Newton’s method with a grid of initial points in the solution space is conjectured to be able to return all globally optimal quaternion solutions for the given measurements.

Truth model simulations for two dynamics scenarios supported the usefulness of the developed algebraic approaches. The numerical implementations of two of these approaches succeeded in correctly solving the nonlinear equations, although additional incorrect solutions were also generated. Additional data resolved the ambiguity by making the problems globally observable. Some combinations of methods and scenarios were found to provide superior numerical convergence, while others were less susceptible to aliasing errors. The failure of two approaches to converge to distinct minimum points illustrated the difficulty of determining whether a given system of equations fully determines a solution.

One possible application of this method is to provide the initial attitude and rate states to a recursive EKF. The global optimality of these initial states should prevent the possibility of EKF divergence. These methods could be applied to data from sounding rocket or simple satellite missions.

Appendix

A. Approach #2 First Ambiguity

This section demonstrates that the first integer ambiguity for Approach #2 will always be zero if no aliasing occurs. With the integer ambiguity included, Eq. (36) can be written:

$$\omega_{ab}(\theta_0, \theta_1, \theta_2) = \pm \frac{2[\arccos(\hat{q}_{01} \cdot \hat{q}_{21}) + 2n_1]}{(t_2 - t_0)}$$  \hspace{1cm} (A1)

where both the arccosine function and the ambiguity $n_1$ are taken to be greater than or equal to zero without loss of generality. It is possible to derive a closed-form expression for the ambiguity $n_1$. First, one rearranges Eq. (A1) to get the ambiguity by itself.

$$n_1 = \pm \frac{\omega_{ab}(t_2 - t_0)}{2(2\pi)} \cdot \frac{\arccos(\hat{q}_{01} \cdot \hat{q}_{21})}{2\pi}$$  \hspace{1cm} (A2)

If $\omega_{ab}$ is negative in Eq. (A1), the minus sign applies in Eq. (A2), and vice versa. As a result, the quantity $\pm \omega_{ab}$ is always positive. For simplicity, assume that the measurement intervals are approximately equal with length $\Delta t$. If the measurements are spaced unevenly, then $\Delta t$ should be taken as the largest measurement interval in order to calculate a worst-case condition. Also, define $\omega_{samp} = 2\pi/\Delta t$ as it was defined for the non-aliasing conditions in Eqs. (20). After making these substitutions, Eq. (A2) becomes

$$n_1 = \pm \frac{\omega_{ab} 2\Delta t}{2(2\pi)} \cdot \frac{\arccos(\hat{q}_{01} \cdot \hat{q}_{21})}{2\pi} = \pm \frac{\omega_{ab}}{\omega_{samp}} \cdot \frac{\arccos(\hat{q}_{01} \cdot \hat{q}_{21})}{2\pi}$$  \hspace{1cm} (A3)
The non-aliasing condition in Eq. (20b) is equivalent to

\[
\frac{|\omega_{ab}|}{\omega_{\text{samp}}} < \frac{1}{2} \tag{A4}
\]

Similarly, the arccosine function can take only values between 0 and \(\pi\). As a result of Eq. (A4) and the limit on the arccosine function, the value of the ambiguity \(n_1\) is restricted.

\[
0 - \frac{1}{2} < n_1 < \frac{1}{2} - 0 \quad \Rightarrow \quad -\frac{1}{2} < n_1 < \frac{1}{2} \tag{A5}
\]

Because the ambiguity \(n_1\) takes only non-negative integer values, Eq. (A5) means that if the non-aliasing condition of Eq. (20b) is satisfied, \(n_1\) is guaranteed to be zero.

**B. Approach #2 Second Ambiguity**

As in Appendix B, the second Approach #2 ambiguity can be shown to be zero when there is no aliasing. This ambiguity appears in the \(\beta\) values of Eq. (43) after determining them with a two-argument arctangent:

\[
\beta = \arctan 2(\pm ||q_{a,3}|| ||q_{b,4}||) + 2\pi n_2 \tag{A6}
\]

where \(n_2\) is an integer which is not necessarily positive, \(q_a\) is the quaternion rotation about the angular momentum axis for any specified time interval, and \(\beta\) is given by Eq. (18) as

\[
\beta = \frac{\Delta t}{2} \frac{||h_n||}{I_r} = \frac{\Delta t}{2} \omega_{as} \tag{A7}
\]

Next, one rearranges Eq. (A6) to get the arctangent function by itself.

\[
\beta - 2\pi n_2 = \arctan 2(\pm ||q_{a,3}|| ||q_{b,4}||) \tag{A8}
\]

The arctangent function takes values on a range from \(-\pi\) to \(\pi\), and this fact can be used to limit the magnitude of the left-hand quantity:

\[
|\beta - 2\pi n_2| \leq \pi \tag{A9}
\]

After dividing all quantities by \(2\pi\) and substituting the expression in Eq. (A7), Eq. (A9) becomes

\[
\frac{\omega_{as} \Delta t}{2(2\pi)} - n_2 \leq \frac{1}{2} \tag{A10}
\]

Equation (A10) can be simplified further by using the notation of the non-aliasing conditions and defining \(\omega_{\text{samp}} = \frac{2\pi}{\Delta t}\). If the time intervals between measurements are equal, then this quantity is the sampling frequency. Otherwise, if \(\Delta t\) represents the longest interval between any two consecutive measurements, \(\omega_{\text{samp}}\) is effectively a worst-case sampling frequency. With this substitution into Eq. (A10),

\[
\frac{\omega_{as}}{2\omega_{\text{samp}}} - n_2 \leq \frac{1}{2} \tag{A11}
\]
Equation (A11) states that the difference between the integer \( n \) and the quantity \( \omega_n/(2\omega_{\text{samp}}) \) is less than 0.5. Because \( n \) can take only integer values, Eq. (A11) can be re-written as

\[
n_z = \text{round}\left( \frac{\omega_n}{2\omega_{\text{samp}}} \right)
\]  

(A12)

where “round” is understood to represent an operation rounding its argument to the nearest integer. Next, consider the non-aliasing condition in Eq. (20a), which can be rearranged as:

\[
\left| \frac{\omega_n}{\omega_{\text{samp}}} \right| < \frac{1}{2}
\]  

(A13)

By extension,

\[
\left| \frac{\omega_n}{2\omega_{\text{samp}}} \right| < \frac{1}{4}
\]  

(A14)

The combination of (A12) and (A14) gives the final result:

\[
\text{round}\left( -\frac{1}{4} \right) \leq n_z \leq \text{round}\left( \frac{1}{4} \right) \implies n_z = 0
\]  

(A15)

References